Dimension, multiplicity, holonomic modules, and an analogue of the inequality of Bernstein for rings of differential operators in prime characteristic

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Dedicated to Joseph Bernstein on the ocassion of his 60th birthday

Abstract

Let K be an *arbitrary* field of characteristic $p > 0$ and $\mathcal{D}(P_n)$ be the ring of differential operators on a polynomial algebra P_n in n variables. A long anticipated *analogue of the inequality of Bernstein* is proved for the ring $\mathcal{D}(P_n)$. In fact, *three different proofs* are given of this inequality (two of which are essentially characteristic free): the first one is based on the concept of the *filter dimension*, the second - on the concept of a *set of holonomic subalgebras with multiplicity*, and the third works only for *finitely presented* modules and follows from a description of these modules (obtained in the paper). On the way, analogues of the concepts of (Gelfand-Kirillov) *dimension, multiplicity, holonomic modules* are found in prime characteristic (giving answers to old questions of finding such analogs). An idea is very simple - to find characteristic free generalizations (and proofs) which in characteristic zero give known results and in prime characteristic - generalizations. An analogue of the *Quillen's Lemma* is proved for simple *finitely presented* $\mathcal{D}(P_n)$-modules. Moreover, for each such module L, $\mathrm{End}_{\mathcal{D}(P_n)}(L)$ is a finite *separable field* extension of K and $\dim_K(\mathrm{End}_{\mathcal{D}(P_n)}(L))$ is equal to the *multiplicity* $e(L)$ of L. In contrast to the characteristic zero case where the Geland-Kirillov dimension of a nonzero finitely generated $\mathcal{D}(P_n)$-module M can be *any natural* number from the interval $[n, 2n]$, in the prime characteristic, the (new) dimension $\mathrm{Dim}(M)$ can be *any real* number from the interval $[n, 2n]$. It is proved that *every holonomic* module has *finite length* but in contrast to the characteristic zero case it is *not true* neither that a nonzero finitely generated module of dimension n is holonomic nor that a holonomic module is *finitely presented*. Some of the surprising results are (i) *each simple finitely presented* $\mathcal{D}(P_n)$-module M is *holonomic* having the multiplicity which is a *natural* number (in characteristic zero rather the *opposite* is true, i.e. $\mathrm{GK}\,(M) = 2n-1$, as a rule), (ii) the dimension $\mathrm{Dim}(M)$ of a nonzero *finitely presented* $\mathcal{D}(P_n)$-module M can be *any natural* number from the interval $[n, 2n]$, (iii) the multiplicity $e(M)$ *exists* for each *finitely presented* $\mathcal{D}(P_n)$-module M and $e(M) \in \mathbb{Q}$, the multiplicity $e(M)$ is a *natural number* if $\mathrm{Dim}(M) = n$, and can be *arbitrary small* rational number if $\mathrm{Dim}(M) > n$.

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Contents

1 Introduction

Throughout the paper, K is a field, $P_n = K[x_1, \ldots, x_n]$ a polynomial algebra in n variables over the field K, a module means a *left* module, $\otimes = \otimes_K$, GK stands for the Gelfand-Kirillov dimension.

In characteristic zero, the ring $\mathcal{D}(P_n)$ of differential operators on P_n (so-called, the *Weyl algebra*) has pleasant properties: it is a simple finitely generated Noetherian domain of Gelfand-Kirillov dimension $\mathrm{GK}\,(\mathcal{D}(P_n)) = 2n$ equipped with a standard filtration such that the associated graded algebra $\mathrm{gr}\,\mathcal{D}(P_n)$ is an affine commutative algebra. *None* of these properties, *except simplicity*, holds for the ring $\mathcal{D}(P_n)$ in prime characteristic. Moreover, in prime characteristic the ring $\mathcal{D}(P_n)$ has a lot of nilpotent elements and zero divisors. This has a serious implication that the standard approach of studying $\mathcal{D}(P_n)$-modules via reduction to modules over affine commutative algebras simply is *not available*.

Key ingredients of the theory of (algebraic) $\mathcal{D}$-modules in characteristic zero are the Gelfand-Kirillov dimension, multiplicity, Hilbert polynomial, the inequality of Bernstein, and holonomic modules. In prime characteristic, straightforward generalizations of these either do not exist or give 'wrong' answers (as in the case of the Gelfand-Kirillov dimension: $\mathrm{GK}\,(\mathcal{D}(P_n)) = n$ in prime characteristic rather than $2n$ as it 'should' be and it is in characteristic zero).

In 70'th and 80'th, for rings of differential operators in prime characteristic natural questions were posed (see, for example, questions 1-4 in [16]) [some of them are still open] that can be summarized as *to find generalizations of the mentioned concepts and results (that results in 'good theory'* expectation of which was/is high, see, the remark of

Björk in [16]). One of the question in [16] is *to give a definition of holonomic module in prime characteristic*. In characteristic zero, holonomic modules have remarkable *homological* properties based on which Mebkhout and Narvaez-Macarro [12] gave a definition of holonomic module. Another approach (based on the *Cartier Lemma*) was taken by Bogvad [6] who defined, so-called, *filtration holonomic* modules. This one is more close to the original idea of holonomicity in characteristic zero. Note that the two mentioned concepts of holonomicity in prime characteristic appeared *before* analogues of the Gelfand-Kirillov dimension and the inequality of Bernstein have been found.

In the present paper, analogues of the Gelfand-Kirillov dimension, multiplicity, the inequality of Bernstein, and holonomic modules are found in prime characteristic based on a simple idea - to find characteristic free generalizations (and proofs) which in characteristic zero give known concepts (and proofs) and in prime characteristic - generalizations.

Filtrations of standard type and the dimension Dim. A part of the success story in studying various finitely generated (Noetherian) algebras is the class of finite dimensional filtrations that are equivalent to standard filtrations (a standard filtration is determined in the obvious way by a finite set of algebra generators). In general, for an algebra which *not finitely generated* (like $\mathcal{D}(P_n)$ in prime characteristic) there is *no obvious* choice of finite dimensional filtrations but for the algebra $\mathcal{D}(P_n)$ there is an obvious one - filtrations that 'correspond' to standard filtrations in characteristic zero, in the present paper they are called *filtrations of standard type* and an analogue of the *Bernstein filtration* is called the *canonical filtration* $F = \{F_i\}_{i\geq 0}$ on $\mathcal{D}(P_n)$ and $\dim_K(F_i) = \binom{i+2n}{2n} = \frac{1}{(2n)!}i^{2n} + \cdots$, $i \geq 0$. Now, a finitely generated $\mathcal{D}(P_n)$-module $M = \mathcal{D}(P_n)M_0$ ($\dim_K(M_0) < \infty$) is equipped with the *filtration of standard type* $\{M_i := F_iM_0\}$ and one can define the *dimension* of M: $\mathrm{Dim}(M) := \gamma(i \mapsto \dim_K(M_i))$ where γ denotes the 'growth' of function. In particular, $\mathrm{Dim}(\mathcal{D}(P_n)) = 2n$.

An analogue of the inequality of Bernstein.

Theorem 1.1 *(***The inequality of Bernstein***, [7]) Let K a field of characteristic zero. Then $\mathrm{GK}\,(M) \geq n$ for all nonzero finitely generated $\mathcal{D}(P_n)$-modules M.*

An analogue of this inequality exists for an arbitrary simple finitely generated algebra.

Theorem 1.2 *[2] Let A be a simple finitely generated algebra. Then*

$$\mathrm{GK}\,(M) \geq \frac{\mathrm{GK}\,(A)}{\mathrm{d}(A) + \max\{\mathrm{d}(A), 1\}}$$

for all nonzero finitely generated A-modules M where $d(A)$ is a (left) filter dimension of A.

In particular, $d(\mathcal{D}(P_n)) = 1$ (see [3]) and $\mathrm{GK}\,(\mathcal{D}(P_n)) = 2n$ in characteristic zero, and so $\mathrm{GK}\,(M) \geq \frac{2n}{1+1} = n$ (Theorem 1.1).

In Section 3, a generalization of Theorem 1.2 (Theorem 3.1) is given for a simple (not necessarily finitely generated or Noetherian) algebra equipped with a finite dimensional filtration.

Theorem 1.3 *Let A be a simple algebra with a finite dimensional filtration $F = \{A_i\}$. Then*

$$\mathrm{Dim}(M) \geq \frac{\mathrm{Dim}(A)}{\mathrm{d}(A) + \max\{\mathrm{d}(A), 1\}}$$

for all nonzero finitely generated A-modules M where d is the filter dimension.

Applying this result to the algebra $\mathcal{D}(P_n)$ in prime characteristic one obtains an analogue of the inequality of Bernstein in prime characteristic.

Theorem 1.4 *Let K a field of characteristic $p > 0$. Then $\mathrm{Dim}(M) \geq n$ for all nonzero finitely generated $\mathcal{D}(P_n)$-modules M.*

Proof. Since $\mathrm{Dim}(\mathcal{D}(P_n)) = 2n$ and $d(\mathcal{D}(P_n)) = 1$ (Theorem 4.2), applying Theorem 1.3 we have $\mathrm{Dim}(M) \geq \frac{2n}{1+1} = n$. $\square$

The proof is essentially characteristic free.

In characteristic zero, the Gelfand-Kirillov dimension of a nonzero finitely generated $\mathcal{D}(P_n)$-module can be *any natural* number from the interval $[n, 2n]$.

Theorem 1.5 *Let K be a field of characteristic $p > 0$.*

1. *(Theorem 9.11) For each <u>real</u> number d from the interval $[n, 2n]$ there exists a cyclic $\mathcal{D}(P_n)$-module M such that $\mathrm{Dim}(M) = d$.*

2. *(Theorem 5.5) The dimension $\mathrm{Dim}(N)$ of a nonzero finitely presented $\mathcal{D}(P_n)$-module N can be any <u>natural</u> number from the interval $[n, 2n]$.*

Holonomic modules. A function $f : \mathbb{N} \to \mathbb{N}$ has *polynomial growth* if there exists a polynomial $p(t) \in \mathbb{Q}[t]$ such that $f(i) \leq p(i)$ for $i \gg 0$. In characteristic zero, a nonzero finitely generated $\mathcal{D}(P_n)$-module M is *holonomic* iff $\mathrm{GK}\,(M) = n$ iff the function $i \mapsto \dim_K(M_i)$ has polynomial growth of degree n (i.e. $\dim_K(M_i) \leq p(i)$ for $i \gg 0$, and $\deg_t(p(t)) = n$) for some/any standard filtration $\{M_i\}$ on M.

Definition. In prime characteristic, a nonzero finitely generated $\mathcal{D}(P_n)$-module M is *holonomic* iff the function $i \mapsto \dim_K(M_i)$ has polynomial growth of degree n (i.e. $\dim_K(M_i) \leq p(i)$ for $i \gg 0$, and $\deg_t(p(t)) = n$) for some (then any) *filtration of standard type* $\{M_i\}$ on M.

- (Proposition 9.9) *In prime characteristic, there exists a cyclic non-holonomic non-Noetherian $\mathcal{D}(P_n)$-module M with $\mathrm{Dim}(M) = n$.*

- (Theorem 9.6) *In prime characteristic, each holonomic module has finite length and it doest not exceed its 'multiplicity'.*

These two results show that even having the analogue of the Gelfand-Kirillov dimension and the analogue of the inequality of Bernstein the 'straightforward' generalization of holonomicity (namely, $\mathrm{Dim}(M) = n$) simply is *not correct.*

Holonomic sets of subalgebras with multiplicity. For a *simple* algebra A (which is not necessarily finitely generated or Noetherian), existence of *holonomic set of subalgebras with multiplicity* is another reason why an analogue of the inequality of Bernstein holds and why each holonomic A-module has finite length (Theorem 9.2).

- (Theorems 9.5 and 9.3). *In prime characteristic, the algebra $\mathcal{D}(P_n)$ has a holonomic set of subalgebras with multiplicity* 1 *(given explicitly).*

Definition. In prime characteristic, a set $\mathcal{C} = \{C_\nu\}_{\nu\in\mathcal{N}}$ of subalgebras of the algebra $\mathcal{D}(P_n)$ is called a *holonomic set of subalgebras with mulitplicity* e if for each nonzero $\mathcal{D}(P_n)$-module M there exists a nonzero finite dimensional vector subspace V of M such that

$$\dim_K(C_{\nu,i}V) \geq \frac{e}{n!}i^n + \cdots , \quad i \gg 0,$$

for some $\nu \in \mathcal{N}$ where $\{C_{\nu,i} := C_\nu \cap F_i\}$ is the induced filtration on the algebra C_ν from the canonical filtration $F = \{F_i\}$ on the algebra $\mathcal{D}(P_n)$ and the three dots mean $o(i^n)$, smaller terms.

Finitely presented $\mathcal{D}(P_n)$-modules and multiplicity. Briefly, in prime characteristic *finitely presented* $\mathcal{D}(P_n)$-modules behave similarly as *finitely generated* $\mathcal{D}(P_n)$-modules in characteristic zero (Theorem 5.10): for *each finitely presented* $\mathcal{D}(P_n)$-module M, the Poincare series of it is a *rational* function, though its Hilbert function is not a polynomial but an *almost polynomial* degree of which coincides with the dimension $\mathrm{Dim}(M)$ of M (and it can be *any natural number* from the interval $[n, 2n]$, this gives another proof of an analogue of the inequality of Bernstein for finitely presented $\mathcal{D}(P_n)$-modules, Theorem 5.5), and the multiplicity exits for M (Theorem 5.5). The differences are (i) in prime characteristic, finitely presented $\mathcal{D}(P_n)$-modules have transparent structure and are described by Theorem 5.5, but in characteristic zero the category of finitely generated $\mathcal{D}(P_n)$-modules is still a mystery, (ii) for each *natural* number d such that $n < d \leq 2n$, there exists a cyclic finitely presented $\mathcal{D}(P_n)$-module M with $\mathrm{Dim}(M) = d$ and with *arbitrary small* multiplicity $e(M)$, Lemma 5.6 (in characteristic zero, multiplicity is a *natural* number), though the multiplicity of *every holonomic finitely presented* $\mathcal{D}(P_n)$-module is a *natural* number (Theorem 8.7), (iii) and what is completely unexpected is that *each simple finitely presented* $\mathcal{D}(P_n)$-module is *holonomic* (Corollary 5.8), and if, in addition, the field K is *algebraically closed* then the multiplicity is *always* 1 (Corollary 6.8).

A classification of simple finitely presented $\mathcal{D}(P_n)$-modules and an analogue of Quillen's Lemma. *In prime characteristic (see Theorem 6.7),*

- *A clasification of simple finitely presented $\mathcal{D}(P_n)$-modules is obtained.*

- *Every simple finitely presented $\mathcal{D}(P_n)$-module M is holonomic, and*

- (**An analogue of Quillen's Lemma**) *its endomorphism algebra* $\mathrm{End}_{\mathcal{D}(P_n)}(M)$ *is a finite separable field extension of* K*, and*
- $\dim_K(\mathrm{End}_{\mathcal{D}(P_n)}(M)) = e(M)$, *the multiplcity of* M*, and*
- *if, in addition, the field* K *is algebraically closed then always* $e(M) = 1$.

A classification of tiny simple $\mathcal{D}(P_n)$-modules. A classification is obtained of the 'smallest' simple $\mathcal{D}(P_n)$-modules (see Theorems 7.1 and 6.7), they are called *tiny* modules. Theorem 6.7 describes the set of tiny *finitely presented* $\mathcal{D}(P_n)$-modules and Theorem 7.1 classifies the set of tiny *non-finitely presented* $\mathcal{D}(P_n)$-modules. They turned out to be *holonomic* with multiplicities which are *natural* numbers. Briefly, they have the same properties as simple finitely presented $\mathcal{D}(P_n)$-modules.

Results of this paper have been generalized for the ring of differential operators on a *smooth irreducible affine algebraic variety*, [5].

2 Filter dimension of algebras and modules

The filter dimension is one of the key ingredients in the proof of an analogue of the inequality of Bernstein in prime characteristic.

Originally, the filter dimension was defined for any *finitely generated* algebra A and any *finitely generated* A-module. In this section, the concept of filter dimension of algebras and modules will be extended to a class of not necessarily finitely generated algebras.

The concept of growth. Let $\mathcal{F}$ be the set of all functions from the set of natural numbers $\mathbb{N} = \{0, 1, \dots\}$ to itself. For each function $f \in \mathcal{F}$, the non-negative real number or ∞ defined as

$$\gamma(f) := \inf\{r \in \mathbb{R} \mid f(i) \leq i^r \text{ for } i \gg 0\}$$

is called the **degree** of f. The function f has **polynomial growth** if $\gamma(f) < \infty$. Let $f, g, p \in \mathcal{F}$, and $p(i) = p^*(i)$ for $i \gg 0$ where $p^*(t) \in \mathbb{Q}[t]$ (a polynomial algebra with coefficients from the field of rational numbers). Then

$$\gamma(f+g) \leq \max\{\gamma(f), \gamma(g)\}, \qquad \gamma(fg) \leq \gamma(f) + \gamma(g),$$
$$\gamma(p) = \deg_t(p^*(t)), \qquad \gamma(pg) = \gamma(p) + \gamma(g).$$

The equivalence relation on the class of filtrations. Let A be an algebra over an *arbitrary* field K. Recall that a *filtration* $F = \{A_i\}_{i\geq 0}$ of the algebra A is an ascending chain of vector subspaces of A:

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_i \subseteq \cdots, \quad A = \cup_{i\geq 0} A_i, \quad K \subseteq A_0, \quad A_i A_j \subseteq A_{i+j}, \quad i, j \geq 0.$$

The filtration F is a **finite dimensional** filtration (or a **finite** filtration, for short) provided $\dim_K F_i < \infty$ for all $i \geq 0$. Filtrations $F = \{A_i\}$ and $G = \{B_i\}$ on A are called *equivalent* ($F \sim G$) if there exist natural numbers a, b, c, d such that $a > 0$, $c > 0$ and

$$A_i \subseteq B_{ai+b} \quad \text{and} \quad B_i \subseteq A_{ci+d} \quad \text{for } i \gg 0.$$

The equivalent filtrations $F = \{A_i\}$ and $G = \{B_i\}$ on A are called *strongly equivalent* if $a = c = 1$. A similar definition exists for filtrations on modules rather than algebras.

Clearly, this is an equivalence relation on the class of all filtrations of the algebra A. For a filtration F, $\widetilde{F}$ denotes the equivalence class of the filtration F. If one of the inclusions above holds, say the first, we write $F \leq G$.

The Gelfand-Kirillov dimension. If $A = K\langle a_1, \dots, a_s\rangle$ is a *finitely generated* K-algebra. The finite dimensional filtration $F = \{A_i\}$ associated with algebra generators $a_1, \dots, a_s$:

$$A_0 := K \subseteq A_1 := K + \sum_{i=1}^{s} Ka_i \subseteq \cdots \subseteq A_i := A_1^i \subseteq \cdots$$

is called the **standard filtration** for the algebra A. Let $M = AM_0$ be a finitely generated A-module where M_0 is a finite dimensional generating subspace of the A-module M. The finite dimensional filtration $\{M_i := A_iM_0\}$ is called the **standard filtration** for the A-module M. All standard filtrations of an algebra A (or a finitely generated A-module) are equivalent.

Definition. $\mathrm{GK}\,(A) := \gamma(i \mapsto \dim_K(A_i))$ and $\mathrm{GK}\,(M) := \gamma(i \mapsto \dim_K(M_i))$ are called the **Gelfand-Kirillov** dimensions of the algebra A and the A-module M respectively.

It is easy to prove that the Gelfand-Kirillov dimension of the algebra (resp. the module) does not depend on the choice of the standard filtration of the algebra (resp. and the choice of the generating subspace of the module) see [9] for details. This is a direct consequence of the fact that all the standard filtrations are equivalent.

The results we are going to generalize first were proved for *finitely generated* algebras (and their *finitely generated* modules) equipped with standard filtrations. Here we extend results to arbitrary filtrations (mainly finite dimensional) on a not necessarily finitely generated algebras. The results do not depend on a filtration inside its equivalence class, but, in general, they do depend on the equivalence class. The choice of the equivalence class depends on a concrete class of algebras.

Our main motivation is an equivalence class of finite dimensional filtrations on a ring of differential operators $\mathcal{D}(A)$ in prime characteristic that in characteristic zero coincide with the class of all the *standard* filtrations on the algebra $\mathcal{D}(A)$.

The return functions and the (left) filter dimension. Let A be a filtered algebra with a filtration $F = \{A_i\}$, and let $M = AM_0$ be a finitely generated A-module with a *finite dimensional* generating subspace M_0. Then $M = \cup_{i\geq 0} M_i$ is a filtered A-module with the filtration $\{M_i := A_iM_0\}$ which obviously does depend on the filtration F and a generating subspace M_0. When one fixes the filtration F then distinct finite dimensional subspaces of the A-module M give *equivalent* filtrations on the module M.

The next definition appeared in [2] in case of standard filtrations.

Definition. The function $\nu_{F,M_0} : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$,

$$\nu_{F,M_0}(i) := \min\{j \in \mathbb{N} \cup \{\infty\} : \ A_jM_{i,gen} \supseteq M_0 \ \text{ for all } \ M_{i,gen}\}$$

is called the **return function** of the A-module M associated with the filtration $F = \{A_i\}$ of the algebra A and the generating subspace M_0 of the A-module M where $M_{i,gen}$ runs

through *all finite dimensional generating subspaces* for the A-module M such that $M_{i,gen} \subseteq M_i$.

Suppose, in addition, that the algebra A is a *simple* algebra. The **return function** $\nu_F \in \mathcal{F}$ and the **left return function** $\lambda_F \in \mathcal{F}$ for the algebra A with respect to the filtration $F := \{A_i\}$ for the algebra A are defined by the rules:

$$\begin{aligned}
\nu_F(i) &:= \min\{j \in \mathbb{N} \cup \{\infty\} \,|\, 1 \in A_j a A_j \text{ for all } 0 \neq a \in A_i\}, \\
\lambda_F(i) &:= \min\{j \in \mathbb{N} \cup \{\infty\} \,|\, 1 \in A a A_j \text{ for all } 0 \neq a \in A_i\},
\end{aligned}$$

where $A_j a A_j$ is the vector subspace of the algebra A spanned over the field K by the elements xay for all $x, y \in A_j$; and AaA_j is the left ideal of the algebra A generated by the set aA_j. Similarly, the **unit return function** $\nu_F^u \in \mathcal{F}$ and the **left unit return function** $\lambda_F^u \in \mathcal{F}$ are defined (where $U = U(A)$ is the *group of units*, i.e. invertible elements of A):

$$\begin{aligned}
\nu_F^u(i) &:= \min\{j \in \mathbb{N} \cup \{\infty\} \,|\, U(A) \cap A_j a A_j \neq \emptyset \text{ for all } 0 \neq a \in A_i\}, \\
\lambda_F^u(i) &:= \min\{j \in \mathbb{N} \cup \{\infty\} \,|\, U(A) \cap A a A_j \neq \emptyset \text{ for all } 0 \neq a \in A_i\}.
\end{aligned}$$

Clearly,

$$\lambda_F^u(i) \leq \lambda_F(i) \leq \nu_F(i) \text{ and } \lambda_F^u(i) \leq \nu_F^u(i) \leq \nu_F(i) \quad \text{for all} \quad i \geq 0. \tag{1}$$

The next result shows that under a mild restriction the four return functions take only *finite* values. In general, there is no reason to believe that values of the return functions are always finite, but for central simple algebras equipped with an arbitrary finite dimensional filtration this is always the case (see the next lemma). Recall that the centre of a simple algebra is a field.

Lemma 2.1 *Let A be a simple algebra equipped with a finite dimensional filtration $F = \{A_i\}$ such that the centre $Z(A)$ of the algebra A is an algebraic field extension of K. Then the four return functions take finite values.*

Proof. In a view of (1), it suffices to prove the lemma for the return function ν_F, that is $\nu_F(i) < \infty$ for all $i \geq 0$.

The centre $Z = Z(A)$ of the simple algebra A is a field that contains K. Let $\{\omega_j \,|\, j \in J\}$ be a K-basis for the K-vector space Z. Since $\dim_K(A_i) < \infty$, one can find finitely many Z-linearly independent elements, say $a_1, \ldots, a_s$, of A_i such that $A_i \subseteq Za_1 + \cdots + Za_s$. Next, one can find a finite subset, say J', of J such that $A_i \subseteq Va_1 + \cdots + Va_s$ where $V = \sum_{j \in J'} K\omega_j$. The field K' generated over K by the elements ω_j, $j \in J'$, is a finite field extension of K (i.e. $\dim_K(K') < \infty$) since Z/K is algebraic, hence $K' \subseteq A_n$ for some $n \geq 0$. Clearly, $A_i \subseteq K'a_1 + \cdots + K'a_s$.

The A-bimodule ${}_AA_A$ is simple with ring of endomorphisms $\mathrm{End}({}_AA_A) \simeq Z$. By the *Density* Theorem, [13], 12.2, for each integer $1 \leq j \leq s$, there exist elements of the algebra A, say $x_1^j, \ldots, x_m^j, y_1^j, \ldots, y_m^j$, $m = m(j)$, such that for all $1 \leq l \leq s$

$$\sum_{k=1}^{m} x_k^j a_l y_k^j = \delta_{j,l}, \quad \text{the Kronecker delta.}$$

Let us fix a natural number, say $d = d_i$, such that A_d contains all the elements x_k^j, y_k^j, and the field K'. We claim that $\nu_F(i) \leq 2d$. Let $0 \neq a \in A_i$. Then $a = \lambda_1 a_1 + \cdots + \lambda_s a_s$ for some $\lambda_i \in K'$. There exists $\lambda_j \neq 0$. Then $\sum_{k=1}^m \lambda_j^{-1} x_k^j a_j y_k^j = 1$, and $\lambda_j^{-1} x_k^j, y_k^j \in A_{2d}$. This proves the claim and the lemma. $\square$

Remark. If the field K is *uncountable* then automatically the centre $Z(A)$ of a simple finitely generated algebra A is algebraic over K (since A has a countable K-basis and the rational function field $K(x)$ has uncountable basis over K since elements $\frac{1}{x+\lambda}$, $\lambda \in K$, are K-linearly independent).

In what follows we will assume that the four return functions **do not take infinite value**.

Lemma 2.2 *Let A be an algebra equipped with two equivalent filtrations $F = \{A_i\}$ and $G = \{B_i\}$.*

1. *Let M be a finitely generated A-module. Then $\gamma(\nu_{F,M_0}) = \gamma(\nu_{G,N_0})$ for any finite dimensional generating subspaces M_0 and N_0 of the A-module M.*

2. *If, in addition, A is a simple algebra then $\gamma(\nu_F) = \gamma(\nu_G)$, $\gamma(\lambda_F) = \gamma(\lambda_G)$, and $\gamma(\nu_F) = \gamma(\nu_{F\otimes F^o,K})$ where $\nu_{F\otimes F^o,K}$ is the return function of the $A \otimes A^o$-module A and A^o is the opposite algebra to A.*

3. *If, in addition, A is a simple algebra then $\gamma(\nu_F^u) = \gamma(\nu_G^u)$ and $\gamma(\lambda_F^u) = \gamma(\lambda_G^u)$.*

Proof. 1. The module M has two filtrations $\{M_i = A_i M_0\}$ and $\{N_i = B_i N_0\}$. Let $\nu = \nu_{F,M_0}$ and $\mu = \nu_{G,N_0}$.

First, we consider two special cases, then the general case will follow easily from these two. Suppose first that $F = G$. Choose a natural number s such that $M_0 \subseteq N_s$ and $N_0 \subseteq M_s$, then $N_i \subseteq M_{i+s}$ and $M_i \subseteq N_{i+s}$ for all $i \geq 0$. Let $N_{i,gen}$ be any generating subspace for the A-module M such that $N_{i,gen} \subseteq N_i$. Since $M_0 \subseteq A_{\nu(i+s)} N_{i,gen}$ for all $i \geq 0$ and $N_0 \subseteq A_s M_0$, we have $N_0 \subseteq A_{\nu(i+s)+s} N_{i,gen}$, hence $\mu(i) \leq \nu(i+s) + s$ and finally $\gamma(\mu) \leq \gamma(\nu)$. By symmetry, the opposite inequality is true and so $\gamma(\mu) = \gamma(\nu)$.

Now, suppose that $M_0 = N_0$. Since $F \sim G$ one can choose natural numbers a, b, c, d such that $a > 0$, $c > 0$ and

$$A_i \subseteq B_{ai+b} \quad \text{and} \quad B_i \subseteq A_{ci+d} \quad \text{for} \quad i \gg 0.$$

Then $N_i = B_i N_0 \subseteq A_{ci+d} M_0 = M_{ci+d}$ for all $i \geq 0$, hence $N_0 = M_0 \subseteq A_{\nu(ci+d)} N_{i,gen} \subseteq B_{a\nu(ci+d)+b} N_{i,gen}$, therefore $\mu(i) \leq a\nu(ci+d) + b$ for all $i \geq 0$, hence $\gamma(\mu) \leq \gamma(\nu)$. By symmetry, we get the opposite inequality which implies $\gamma(\mu) = \gamma(\nu)$.

In the general case, $\gamma(\nu_{F,M_0}) = \gamma(\nu_{F,N_0}) = \gamma(\nu_{G,N_0})$.

2. The algebra A is simple, equivalently, it is a simple (left) $A \otimes A^o$-module where A^o is the *opposite* algebra to A. The opposite algebra has the filtration $F^o = \{A_i^o\}$. The tensor product of algebras $A \otimes A^o$, so-called, the *enveloping algebra* of A, has the filtration $F \otimes F^o = \{C_n\}$ which is the tensor product of the filtrations F and F^o, that is,

$C_n = \sum\{A_i \otimes A_j^o, i+j \leq n\}$. Let $\nu_{F\otimes F^o, K}$ be the return function of the $A\otimes A^o$-module A associated with the filtration $F\otimes F^o$ and the generating subspace K. Then

$$\nu_F(i) \leq \nu_{F\otimes F^o, K}(i) \leq 2\nu_F(i) \text{ for all } i \geq 0,$$

and so

$$\gamma(\nu_F) = \gamma(\nu_{F\otimes F^o, K}), \tag{2}$$

and, by the first statement, we have $\gamma(\nu_F) = \gamma(\nu_{F\otimes F^o, K}) = \gamma(\nu_{G\otimes G^o, K}) = \gamma(\nu_G)$, as required. Using a similar argument as in the proof of the first statement one can proof that $\gamma(\lambda_F) = \gamma(\lambda_G)$. We leave this as an exercise.

3. Let F, G, a, b, c, d be as above. Let $U = U(A)$ be the group of units of the algebra A, and let $\lambda := \nu_F^u$ and $\mu := \nu_G^u$ (resp. $\lambda := \lambda_F^u$ and $\mu := \lambda_G^u$). We prove two cases simultaneously. Let x be a nonzero element of A_i. Then $0 \neq x \in B_{ai+b}$ and

$$\begin{aligned} \emptyset &\neq U \cap B_{\mu(ai+b)} x B_{\mu(ai+b)} \subseteq U \cap A_{c\mu(ai+b)+d} x A_{c\mu(ai+b)+d}, \\ \emptyset &\neq U \cap AxB_{\mu(ai+b)} \subseteq U \cap AxA_{c\mu(ai+b)+d} \quad \text{respectively.} \end{aligned}$$

In both cases, $\gamma(\lambda) \leq \gamma(c\mu(ai+b)+d) \leq \gamma(\mu)$. By symmetry, the inverse inequality is also true, and so $\gamma(\lambda) = \gamma(\mu)$. $\square$

Definition. $\mathrm{fd}(M) = \gamma(\nu_{F,M_0})$ is the **filter dimension** of the A-module M, and $\mathrm{fd}(A) := \mathrm{fd}({}_{A\otimes A^o}A)$ is the **filter dimension** of the algebra A. If, in addition, the algebra A is simple, then $\mathrm{fd}(A) = \gamma(\nu_F)$, $\mathrm{lfd}(A) := \gamma(\lambda_F)$ is called the **left filter dimension** of the algebra A, $\mathrm{ud}(A) = \gamma(\nu_F^u)$ is called the **unit dimension** of A, and $\mathrm{lud}(A) := \gamma(\lambda_F^u)$ is called the **left unit dimension** of the algebra A.

By the previous lemma, the definitions make sense provided an equivalence class of filtrations is fixed. We will always assume that we have fixed such a class. A particular choice of an equivalence class of filtrations depends on a class of algebras we study. For finitely generated algebras such an equivalence class as a rule is the equivalence class of all standard filtrations, but for algebras that are not finitely generated there is *no obvious choice* of an equivalence class of filtrations.

For standard filtrations the concept of (left) filter dimension first appeared in [2].

By (1),

$$\mathrm{lud}(A) \leq \mathrm{lfd}(A) \leq \mathrm{fd}(A) \quad \text{and} \quad \mathrm{lud}(A) \leq \mathrm{ud}(A) \leq \mathrm{fd}(A). \tag{3}$$

3 Dimension of (not necessarily finitely generated or Noetherian) algebras and dimension of their finitely generated modules

Theorem 3.1 is the main result of this section, it is a kind of the inequality of Bernstein but for an arbitrary simple algebra (not necessarily finitely generated) equipped with a finite dimensional filtration. In this section, let A be an algebra over an *arbitrary* field K with a *finite dimensional* filtration $F = \{A_i\}$. Let $M = AM_0$ be a finitely generated A-module with

a finite dimensional generating subspace M_0. Then M has a finite dimensional filtration $\{M_i := A_i M_0\}$. Suppose that $G = \{B_i\}$ is a finite dimensional filtration on A equivalent to the filtration F and let N_0 be another finite dimensional generating subspace for the A-module M. Then the A-module M has a second finite dimensional filtration $\{N_i := A_i N_0\}$. It follows easily that $\gamma(\dim_K A_i) = \gamma(\dim_K B_i)$ and $\gamma(\dim_K M_i) = \gamma(\dim_K N_i)$.

Definition. The **dimension** $\mathrm{Dim}\, A$ of the algebra A and the **dimension** $\mathrm{Dim}\, M$ of the finitely generated A-module M are the numbers $\gamma(\dim_K A_i)$ and $\gamma(\dim_K M_i)$ respectively.

So, the dimension $\mathrm{Dim}\, A$ of the algebra A is an invariant of the algebra A *and* the equivalence class of the filtration F. The same is true about the dimension $\mathrm{Dim}\, M$ of the A-module M.

If A is a finitely generated algebra and $\{A_i\}$ is a *standard* filtration then $\mathrm{Dim}(A) = \mathrm{GK}\,(A)$ and $\mathrm{Dim}(M) = \mathrm{GK}\,(M)$.

In this paper, $\mathrm{d}(A)$ stands for any of the dimensions $\mathrm{fd}(A)$, $\mathrm{lfd}(A)$, $\mathrm{ud}(A)$ or $\mathrm{lud}(A)$ of an algebra A (i.e. $\mathrm{d} = \mathrm{fd}, \mathrm{lfd}, \mathrm{ud}, \mathrm{lud}$).

The four dimensions appear naturally when one tries to find a *lower* bound for the holonomic number (Theorem 3.1).

The next theorem is a generalization of the **inequality of Bernstein** (Theorem 1.1) to the class of simple algebras. This result was first appeared in [2, 4] in the case of *simple finitely generated* algebras with respect to the class of *standard* filtrations and for $d = \mathrm{fd}, \mathrm{lfd}$.

Theorem 3.1 *Let A be a simple algebra with a finite dimensional filtration $F = \{A_i\}$. Then*

$$\mathrm{Dim}(M) \geq \frac{\mathrm{Dim}(A)}{\mathrm{d}(A) + \max\{\mathrm{d}(A), 1\}}$$

for all nonzero finitely generated A-modules M where $\mathrm{d} = \mathrm{fd}, \mathrm{lfd}, \mathrm{ud}, \mathrm{lud}$.

Proof. In a view of (3), it suffices to prove the theorem for $d = \mathrm{lud}$. Let $\lambda = \lambda_F^u$ be the left unit return function associated with the finite dimensional filtration F of the algebra A and let $0 \neq a \in A_i$. It follows from the inclusion

$$AaM_{\lambda(i)} = AaA_{\lambda(i)}M_0 \supseteq (U(A) \cap AaA_{\lambda(i)})M_0 \neq 0$$

that the linear map

$$A_i \to \mathrm{Hom}_K(M_{\lambda(i)}, M_{\lambda(i)+i}), a \mapsto (m \mapsto am),$$

is injective, and so $\dim\, A_i \leq \dim\, M_{\lambda(i)} \dim\, M_{\lambda(i)+i}$. Using the above elementary properties of the degree (see also [11], 8.1.7), we have

$$\begin{aligned} \mathrm{Dim}(A) &= \gamma(\dim A_i) \leq \gamma(\dim M_{\lambda(i)}) + \gamma(\dim M_{\lambda(i)+i}) \\ &\leq \gamma(\dim M_i)\gamma(\lambda) + \gamma(\dim M_i) \max\{\gamma(\lambda), 1\} \\ &= \mathrm{Dim}(M)(\mathrm{lud}A + \max\{\mathrm{lud}A, 1\}) \\ &\leq \mathrm{Dim}(M)(\mathrm{lud}A + \max\{\mathrm{lud}A, 1\}). \ \square \end{aligned}$$

The *inequality of Bernstein* says that $\mathrm{GK}\,(M) \geq n$ *for any nonzero finitely generated module* M *over a ring of differential operators* $\mathcal{D}(X)$ *on a smooth irreducible affine algebraic variety* X *of dimension* $n = \dim X$ *over a field of characteristic zero*. Since $\mathrm{GK}\,(\mathcal{D}(X)) = 2n$ and $\mathrm{fd}(\mathcal{D}(X)) = \mathrm{lfd}(\mathcal{D}(X)) = 1$ ([3]), by Theorem 3.1, we have a 'short' proof of the inequality of Bernstein:

$$\mathrm{GK}\,(M) \geq \frac{2n}{1+1} = n.$$

Definition. $h_A := \inf\{\mathrm{Dim}(M) \,|\, M$ is a nonzero finitely generated A-module$\}$ is called the **holonomic number** for the algebra A (with respect to the equivalence class $\widetilde{F}$ of the finite dimensional filtration F).

The result above gives a *lower bound* for the holonomic number of the simple algebra A:

$$h_A \geq \frac{\mathrm{GK}\,(A)}{\mathrm{d}(A) + \max\{\mathrm{d}(A), 1\}}.$$

Theorem 3.2 *Let* A *and* $\widetilde{F}$ *be as above. Then*

$$\mathrm{Dim}(M) \leq \mathrm{Dim}(A)\,\mathrm{fd}(M)$$

for any simple A*-module* M*.*

Proof. Let $\nu = \nu_{F,Km}$ be the return function of the module M associated with the finite dimensional filtration $F = \{A_i\}$ of the algebra A and a fixed nonzero element $m \in M$. Let $\pi : M \to K$ be a non-zero linear map satisfying $\pi(m) = 1$. Then, for any $i \geq 0$ and any $0 \neq u \in M_i := A_i m$: $1 = \pi(m) \in \pi(A_{\nu(i)} u)$, and so the linear map

$$M_i \to \mathrm{Hom}_K(A_{\nu(i)}, K),\ u \mapsto (a \mapsto \pi(au)),$$

is an *injective* map hence dim $M_i \leq$ dim $A_{\nu(i)}$ and finally $\mathrm{Dim}(M) \leq \mathrm{Dim}(A)\,\mathrm{fd}(M)$. $\square$

Corollary 3.3 *Let* A *be a simple algebra with* $\mathrm{Dim}(A) > 0$*. Then*

$$\mathrm{fd}(A) \geq \frac{1}{2}.$$

Proof. Clearly, $\mathrm{Dim}(A \otimes A^o) \leq \mathrm{Dim}(A) + \mathrm{Dim}(A^o) = 2\mathrm{Dim}(A)$. Applying Theorem 3.2 to the simple $A \otimes A^o$-module $M = A$ we finish the proof

$$\mathrm{Dim}(A) = \mathrm{Dim}({}_{A\otimes A^o}A) \leq \mathrm{Dim}(A \otimes A^o)\mathrm{fd}({}_{A\otimes A^o}A) \leq 2\mathrm{Dim}(A)\mathrm{fd}(A)$$

hence $\mathrm{fd}(A) \geq \frac{1}{2}$. $\square$

Corollary 3.4 *Let* A *be a simple algebra with* $\mathrm{Dim}(A) > 0$*. Then*

$$\mathrm{fd}(M) \geq \frac{1}{\mathrm{fd}(A) + \max\{\mathrm{fd}(A), 1\}}$$

for all simple A*-modules* M*.*

Proof. Applying Theorem 3.1 and Theorem 3.2, we have the result

$$\mathrm{fd}(M) \geq \frac{\mathrm{Dim}(M)}{\mathrm{Dim}(A)} \geq \frac{\mathrm{Dim}(A)}{\mathrm{Dim}(A)(\mathrm{fd}(A)+\max\{\mathrm{fd}(A),1\})} = \frac{1}{\mathrm{fd}(A)+\max\{\mathrm{fd}(A),1\}}. \quad \square$$

In general, it is difficult to find the exact value for the filter dimension but for the ring of differential operators $\mathcal{D}(P_n)$ with polynomial coefficients $P_n = K[x_1, \ldots, x_n]$ over a field K of characteristic $p > 0$ it is easy and one can find it directly (Theorem 4.2).

4 An analogue of the inequality of Bernstein for the ring of differential operators $\mathcal{D}(P_n)$ with polynomial coefficients

In this section, K is an *arbitrary* field of characteristic $p > 0$, $P_n := K[x_1, \ldots, x_n]$ is a polynomial algebra, $\mathcal{D} = \mathcal{D}(P_n)$ is the ring of differential operators on P_n. In this section, the concepts of *filtration of standard type* and of *holonomic module* are introduced, it is proved that the filter dimension of the ring $\mathcal{D}(P_n)$ is 1 (Theorem 4.2) and an analogue of the inequality of Bernstein is established (Theorem 4.3) in prime characteristic. We start with recalling some facts and properties of higher derivations (Hasse-Schmidt derivations) which will be used freely in the paper.

Higher derivations. Let us recall basic facts about higher derivations. For more detail the reader is referred to [10], Sec. 27.

A sequence $\delta = (1 := \mathrm{id}_A, \delta_1, \delta_2, \ldots)$ of K-linear maps from a commutative K-algebra A to itself (where id_A is the identity map on A) is called a *higher derivation* (or a *Hasse-Schmidt derivation*) over K from A to A if, for each $k \geq 0$,

$$\delta_k(xy) = \sum_{i+j=k} \delta_i(x)\delta_j(y) \quad \text{for all } x, y \in A. \tag{4}$$

Clearly, $\delta_1 \in \mathrm{Der}_K(A)$. These conditions are equivalent to saying that the map $e : A \to A[[t]]$, $x \mapsto \sum_{i\geq 0} \delta_i(x)t^i$, is a K-algebra homomorphism where $A[[t]]$ is a ring of power series with coefficients from A, or equivalently, that the map

$$e : A[[t]] \to A[[t]], \;\; t \mapsto t, \;\; x \mapsto \sum_{i\geq 0} \delta_i(x)t^i \quad (x \in A),$$

is a $K[[t]]$-algebra homomorphism. Clearly, e is a $K[[t]]$-algebra automorphism of $A[[t]]$, and vice versa (any automorphism $e \in \mathrm{Aut}_{K[[t]]}(A[[t]])$ of the type $e(a) = a + \sum_{i\geq 1} \delta_i(a)t^i$ yields a higher derivation (δ_i) where $a \in A$).

The set $\mathrm{HS}_K(A)$ of all higher K-derivations from A to A is a subgroup of the group $\mathrm{Aut}_K(A[[t]])$ of all K-algebra automorphisms of $A[[t]]$. It follows immediately that a higher derivation has a *unique* extension to a localization $S^{-1}A$ of the algebra A at a multiplicative subset S of A.

Let $\mathcal{D}(A)$ be the *ring of differential operators* on the algebra A and let $\{\mathcal{D}(A)_i\}_{i\geq 0}$ be its *order filtration*. Recall that $\mathcal{D}(A) = \cup_{i\geq 0}\mathcal{D}(A)_i \subseteq \mathrm{End}_K(A)$, $\mathcal{D}(A)_0 := \mathrm{End}_A(A) \simeq A$, and

$$\mathcal{D}(A)_i := \{f \in \mathrm{End}_K(A) : \ fx - xf \in \mathcal{D}(A)_{i-1} \ \text{ for all } \ x \in A\}, \ \ i \geq 1.$$

Let $\delta = (\delta_i) \in \mathrm{HS}_K(A)$. By (4),

$$\delta_i \in \mathcal{D}(A)_i, \ \ i \geq 0, \tag{5}$$

since $\delta_i x - x\delta_i = \sum_{j=0}^{i-1} \delta_{i-j}(x)\delta_j$ for all $x \in A$ and the result follows by induction on i. For each $i \geq 1$ and $x \in A$,

$$\sum_{j\geq 0} \delta_j(x^{p^i})t^j = e(x^{p^i}) = e(x)^{p^i} = \sum_{k\geq 0} \delta_k(x)^{p^i} t^{kp^i},$$

and so $\delta_{kp^i}(x^{p^i}) = \delta_k(x)^{p^i}$ for all $i, k \geq 0$ and $x \in A$; and $\delta_j(x^{p^i}) = 0$ for all j such that $p^i \not| j$ (p^i does not divide j). In particular, $\delta_l(KA^{p^{i+1}}) = 0$ for all $i \geq 1$ and $0 < l < p^{i+1}$.

The higher derivations $(1, \frac{\partial}{1!}, \frac{\partial^2}{2!}, \ldots) \in \mathrm{HS}_K(K[x])$ **where** $\partial = \frac{d}{dx}$. Given a polynomial algebra $K[x]$ in a single variable x over K, the K-algebra homomorphism $K[x] \to K[x][[t]]$, $f(x) \mapsto f(x+t) = \sum_{i\geq 0} \frac{\partial^i}{i!}(f)t^i$, determines the higher derivation $(1, \frac{\partial}{1!}, \frac{\partial^2}{2!}, \ldots) \in \mathrm{HS}_K(K[x])$. If $\mathrm{char}(K) = 0$ then $\frac{\partial^j}{j!}$ means $(j!)^{-1}\partial^j$, but if $\mathrm{char}(K) = p > 0$ then

$$\frac{\partial^j}{j!}(x^i) = \binom{i}{j} x^{i-j} \tag{6}$$

where $\binom{i}{j}$ is the binomial in characteristic p: $\binom{i}{j} = 0$ if $j > i$, and for $j \leq i$, let $j = \sum j_k p^k$, $0 \leq j_k < p$, and $i = \sum i_k p^k$, $0 \leq i_k < p$. Then

$$\binom{i}{j} = \prod_k \binom{i_k}{j_k} \tag{7}$$

where $\binom{i_k}{j_k} = 0$ if $j_k > i_k$, and $\binom{i_k}{j_k} = \frac{i_k!}{j_k!(i_k-j_k)!}$ if $j_k \leq i_k$. The formulas (6) and (7) are obvious when one looks at the following product:

$$(x+t)^i = \prod_k (x^{p^k} + t^{p^k})^{i_k} = \prod_k \sum_{l_k=0}^{i_k} \binom{i_k}{l_k} x^{(i_k-l_k)p^k} t^{l_k p^k} = \sum \prod_k \binom{i_k}{i_k - l_k} x^{\sum_\nu (i_\nu - l_\nu)p^\nu} t^{\sum_\mu l_\mu p^\mu}$$

where the sum runs through all $l_0, l_1, \ldots$ that satisfy $0 \leq l_0 \leq i_0, 0 \leq l_1 \leq i_1, \ldots$. The binomials in characteristic $p > 0$ has a remarkable property - the *translation invariance* (with respect to the p-*adic scale*):

$$\binom{p^k i}{p^k j} = \binom{i}{j}, \quad k \geq 0. \tag{8}$$

This follows directly from (7). By (7), $\binom{i}{j}\neq 0$ iff $i_k\geq j_k$ for all k. It follows that $\binom{pi}{(p-1)i}=0$ for all $i\geq 0$.

Remark. Though $\partial^p=0$ but $\frac{\partial^p}{p!}\neq 0$ since $\frac{\partial^p}{p!}(x^p)=1$ and $\frac{\partial^p}{p!}$ is *not* a derivation as $\frac{\partial^p}{p!}(x)x^{p-1}+x\frac{\partial^p}{p!}(x^{p-1})=0$ (recall that if δ a derivation then so is δ^p).

A higher derivation $\delta=(\delta_i)\in \mathrm{HS}_K(A)$ is called *iterative* if $\delta_i\delta_j=\binom{i+j}{i}\delta_{i+j}$ for all $i,j\geq 0$. Then a direct computation shows that

$$\delta_i^p=0 \quad \text{for all} \quad i\geq 1, \tag{9}$$

$\delta_i^p=\delta_i\cdots\delta_i=\binom{2i}{i}\binom{3i}{2i}\cdots\binom{pi}{(p-1)i}\delta_{pi}=0\delta_{pi}=0$. For $i=1$, we have $\delta_1^p=0$. The higher derivation $(\frac{\partial^i}{i!})\in \mathrm{HS}_K(K[x])$ is iterative as follows directly from the definition of $(\frac{\partial^i}{i!})$.

Given $\delta\in \mathrm{Der}_K(A)$, then $\delta^p\in\mathrm{Der}_K(A)$ and, for any $a\in A$, $(a\delta)^p=a^p\delta^p+(a\delta)^{p-1}(a)\delta$ (the *Hochschild's formula*, [10], 25.5). In the algebra $\mathcal{D}(P_n)$, for each $i=1,\dots,n$, $\partial_i^p=0$, and therefore,

$$(x_i\partial_i)^p=x_i\partial_i. \tag{10}$$

The higher derivations $(1,\frac{\partial_i}{1!},\frac{\partial_i^2}{2!},\dots)\in \mathrm{HS}_K(P_n)$**,** $i=1,\dots,n$. The K-algebra homomorphism $P_n\to P_n[[t]]$,

$$f(x_1,\dots,x_n)\mapsto f(x_1,\dots,x_{i-1},x_i+t,x_{i+1},\dots,x_n)=\sum_{i\geq 0}\frac{\partial_i^k}{k!}(f)t^k,$$

gives the higher derivation $(1,\frac{\partial_i}{1!},\frac{\partial_i^2}{2!},\dots)\in \mathrm{HS}_K(P_n)$. If $\mathrm{char}(K)=0$ then $\frac{\partial_i^k}{k!}$ means $(k!)^{-1}\partial_i^k$, but if $\mathrm{char}(K)=p>0$ then repeating the proof of (6), we see that

$$\frac{\partial_i^k}{k!}(x_j^l)=\delta_{ij}\binom{l}{k}x_j^{l-k} \tag{11}$$

for all $l\geq k\geq 1$ and $1\leq i,j\leq n$ where δ_{ij} is the Kronecker delta; and $\frac{\partial_i^k}{k!}(x_j^l)=0$ if $k>l$.

For an ideal I of the polynomial algebra P_n, $I[[t]]$ is an ideal of the algebra $P_n[[t]]$, and the factor algebra $P_n[[t]]/I[[t]]\simeq P_n/I[[t]]$. The set $\mathrm{HS}_K(P_n,I):=\{e\in \mathrm{HS}_K(P_n)\,|\,e(I[[t]])=I[[t]]\}$ is a subgroup of the group $\mathrm{HS}_K(P_n)$. Note that

$$e(I[[t]])=I[[t]]\Leftrightarrow e(I[[t]])\subseteq I[[t]]. \tag{12}$$

The implication $(\Rightarrow)$ is obvious. The reverse implication follows immediately from the fact that, for any K-algebra A and any higher derivation $\delta=(1,\delta_1,\dots)\in\mathrm{HS}_K(A)$, the inverse automorphism to the automorphism $e(a):=\sum\delta_i(a)t^i$ has the form

$$e^{-1}(a)=a+\delta_1'(a)t+\cdots+\delta_i'(a)t^i+\cdots \tag{13}$$

where $\delta_i'=\sum\pm\omega_{ij}$ is a finite sum where ω_{ij} is a product of certain δ_k'th. Note that $e(I)\subseteq I[[t]]$ iff $\delta_i(I)\subseteq I$ for all $i\geq 1$ where $e(p)=\sum\delta_i(p)t^i$. Then the inclusion $e(I)\subseteq I[[t]]$ implies the inclusions $e^{-1}(I)\subseteq e^{-1}(I[[t]])\subseteq I[[t]]$. Therefore, $e^{\pm 1}(I[[t]])\subseteq I[[t]]$, and

so $e(I[[t]]) = I[[t]]$. Therefore, $\mathrm{HS}_K(P_n, I) := \{e \in \mathrm{HS}_K(P_n) \,|\, e(I[[t]]) \subseteq I[[t]]\}$. Then it follows that the set $\mathrm{hs}_K(P_n, I) := \{e \in \mathrm{HS}_K(P_n) \,|\, e(P_n) \subseteq P_n + I[[t]]\}$ is a normal subgroup of $\mathrm{HS}_K(P_n, I)$. The kernel of the canonical homomorphism of groups

$$\mathrm{HS}_K(P_n, I) \to \mathrm{HS}_K(P_n/I), \quad e \mapsto (p + I \mapsto e(p) + I[[t]]), \tag{14}$$

is equal to $\mathrm{hs}_K(P_n, I)$, and so the map in the proposition is a group monomorphism.

Proposition 4.1 *The map*

$$\mathrm{HS}_K(P_n, I)/\mathrm{hs}_K(P_n, I) \to \mathrm{HS}_K(P_n/I), \quad e \cdot \mathrm{hs}_K(P_n, I) \mapsto (p + I \mapsto e(p) + I[[t]]),$$

is an isomorphism of groups.

Proof. It remains to show that the map is surjective. Given $\overline{e} \in \mathrm{HS}_K(P_n/I)$. For each $i = 1, \ldots, n$, $\overline{e}(x_i + I) = x_i + I + \sum_{j\geq 1}(p_{ij} + I)t^j$ for some $p_{ij} \in P_n$. The automorphism $\overline{e}$ can be extended to an element $e \in \mathrm{HS}_K(P_n)$ setting $e(x_i) = x_i + \sum_{j\geq 1} p_{ij}t^j$ such that the element $\overline{e}$ is the image of the element e under the homomorphism (14). This proves the surjectivity. $\square$

Suppose, for a moment, that $\mathrm{char}(K) = 0$. Then the ring of differential operators $\mathcal{D}(P_n)$ is, so-called, the **Weyl algebra** $A_n = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$, has the *standard* filtration $\{A_{n,i} = \bigoplus_{|\alpha|+|\beta|\leq i} Kx^\alpha \partial^\beta\}_{i\geq 0}$ associated with the set of canonical generators x_j, $\partial_j := \frac{\partial}{\partial x_j}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\partial^\beta := \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

The polynomial algebra P_n as the left A_n-module has the *standard* filtration $\{P_{n,i} := A_{n,i}K = \bigoplus_{|\alpha|\leq i} Kx^\alpha\}$. For each $i \geq 0$,

$$\dim A_{n,i} = \binom{i+2n}{2n} = \frac{(i+2n)(i+2n-1)\cdots(i+1)}{(2n)!} \quad \text{and} \quad \dim P_{n,i} = \binom{i+n}{n},$$

and so $\mathrm{GK}\,(A_n) = 2n$ and $\mathrm{GK}\,(P_n) = n$. The associated graded algebra

$$\mathrm{gr}(A_n) := \oplus_{i\geq 0} A_{n,i}/A_{n,i-1} = K[\overline{x}_1, \ldots, \overline{x}_n, \overline{\partial}_1, \ldots, \overline{\partial}_n]$$

is isomorphic as a graded algebra to a polynomial algebra in $2n$ variables with usual grading.

If $\mathrm{char}(K) = p > 0$ then the ring of differential operators $\mathcal{D} := \mathcal{D}(P_n)$ on P_n is an algebra generated by $x_1, \ldots, x_n$ and commuting *higher derivations* $\frac{\partial_i^k}{k!}$, $i = 1, \ldots, n$ and $k \geq 1$ that satisfy the following *defining* relations:

$$[x_i, x_j] = [\frac{\partial_i^k}{k!}, \frac{\partial_j^l}{l!}] = 0, \quad \frac{\partial_i^k}{k!}\frac{\partial_i^l}{l!} = \binom{k+l}{k}\frac{\partial_i^{k+l}}{(k+l)!}, \quad [\frac{\partial_i^k}{k!}, x_j] = \delta_{ij}\frac{\partial_i^{k-1}}{(k-1)!}, \tag{15}$$

for all $i, j = 1, \ldots, n$ and $k, l \geq 1$ where δ_{ij} is the Kronecker delta and $\frac{\partial_i^0}{0!} := 1$. We will use also the following notation: $\partial_i^{[k]} := \frac{\partial_i^k}{k!}$.

The involution $*$. The K-linear map $*:\mathcal{D}\rightarrow\mathcal{D}$, $x_i\mapsto x_i$, $\partial_i^{[j]}\mapsto(-1)^j\partial_i^{[j]}$, $i=1,\ldots,n$, $j\geq 1$, is an *involution* of the algebra $\mathcal{D}$ ($a^{**}=a$ and $(ab)^*=b^*a^*$). So, the algebra $\mathcal{D}$ is a symmetric object, its 'left' and 'right' properties are the 'same'. In particular, the categories of left and right $\mathcal{D}$-modules are 'identical'.

The $\mathcal{D}(P_n)$-module P_n. The polynomial algebra P_n is a (left) $\mathrm{End}_K(P_n)$-module, $\mathcal{D}(P_n)$ is a subalgebra of $\mathrm{End}_K(P_n)$, and so P_n is a (left) $\mathcal{D}(P_n)$-module. The $\mathcal{D}(P_n)$-module P_n is canonically isomorphic to the factor module $\mathcal{D}(P_n)/\sum_{0\neq\beta\in\mathbb{N}^n}\mathcal{D}(P_n)\frac{\partial^\beta}{\beta!}$.

The algebra $\mathcal{D}$ is not finitely generated and not a (left or right) Noetherian. It follows from the relations that $F=\{F_i:=\bigoplus_{|\alpha|+|\beta|\leq i}Kx^\alpha\frac{\partial^\beta}{\beta!}\}$ is a finite dimensional filtration for the algebra $\mathcal{D}$ where $\frac{\partial^\beta}{\beta!}:=\frac{\partial_1^{\beta_1}}{\beta_1!}\cdots\frac{\partial_n^{\beta_n}}{\beta_n!}$.

Definition. The filtration $F=\{F_i\}$ is called the **canonical filtration** on $\mathcal{D}(P_n)$. If $M=\mathcal{D}M_0$ ($\dim_K(M_0)<\infty$) is a finitely generated $\mathcal{D}$-module then the finite dimensional filtration $\{M_i:=F_iM_0\}_{i\geq 0}$ is called the **canonical filtration** of M.

In characteristic zero, this filtration coincide with the standard filtration $\{A_{n,i}\}$. The canonical filtration is $*$-invariant: $F^*=F$, i.e. $F_i^*=F_i$ for all $i\geq 0$.

Definition. A (finite dimensional) filtration $\{F_i'\}_{i\geq 0}$ on the algebra $\mathcal{D}$ which is *equivalent* to the *canonical* filtration F is called a **filtration of standard type** of $\mathcal{D}$. If $M=\mathcal{D}M_0$ ($\dim_K(M_0)<\infty$) is a finitely generated $\mathcal{D}$-module then the finite dimensional filtration $\{F_i'M_0\}$ is called a **filtration of standard type** of M.

Filtrations of standard type in prime characteristic are correct generalizations of standard filtrations in zero characteristic. Each canonical filtration is a filtration of standard type.

The polynomial algebra P_n as a left $\mathcal{D}$-module has the filtration of standard type $\{F_iK=P_{n,i}\}$. Since $\dim F_i=\dim A_{n,i}$ and $\dim F_iK=\dim P_{n,i}$,

$$\mathrm{Dim}(\mathcal{D}(P_n))=2n\quad\text{and}\quad\mathrm{Dim}(P_n)=\mathrm{GK}\,(P_n)=n.$$

Note that the Gelfand-Kirillov dimension $\mathrm{GK}\,(\mathcal{D}(P_n))=n$, not $2n$. The associated graded algebra $\mathrm{gr}\,\mathcal{D}:=\oplus_{i\geq 0}F_i/F_{i-1}$ ($F_{-1}:=0$) is a commutative algebra which is not a finitely generated algebra, the nil-radical $\mathfrak{n}$ of the algebra $\mathrm{gr}\,\mathcal{D}$ is equal to $\sum_{\alpha,\beta\in\mathbb{N}^n,|\beta|>0}K\overline{x}^\alpha\frac{\overline{\partial}^\beta}{\beta!}$ and $\mathrm{gr}(\mathcal{D})/\mathfrak{n}\simeq K[\overline{x}_1,\ldots,\overline{x}_n]$ is a polynomial algebra.

Theorem 4.2 *Let $\mathcal{D}(P_n)$ be the ring of differential operators with polynomial coefficients $P_n=K[x_1,\ldots,x_n]$ over a field K of characteristic $p>0$. Then $\mathrm{d}(\mathcal{D}(P_n))=1$ where $\mathrm{d}=\mathrm{fd},\mathrm{lfd},\mathrm{ud},\mathrm{lud}$.*

Proof. Let $\nu=\nu_F$ be the return function of the algebra $\mathcal{D}=\mathcal{D}(P_n)$ associated with the canonical filtration $F=\{F_i\}$ on $\mathcal{D}$. Let us prove that $\nu(i)\leq i$ for all $i\geq 0$. We use induction on i. The case $i=0$ is obvious as $F_0=K$. Suppose that $i>0$ and the statement is true for all $i'<i$. Let $a\in F_i\backslash F_{i-1}$. Then $a=\sum a_{\alpha\beta}x^\alpha\frac{\partial^\beta}{\beta!}$ with $|\alpha|+|\beta|\leq i$ and $a_{\alpha\beta}\in K$. If there exists a coefficient $a_{\alpha\beta}\neq 0$ for some $\beta\neq 0$, i.e. $\beta_j\neq 0$ for some j, then applying the inner derivation $\mathrm{ad}\,x_j$ of the algebra $\mathcal{D}$ to the element a we have a *nonzero* element $x_ja-ax_j\in F_{i-1}$, then induction gives the result.

Now, we have to consider the case where $a_{\alpha\beta} = 0$ for all $\beta \neq 0$, that is $a \in P_{n,i} \backslash P_{n,i-1}$. Then there exists a variable, say x_j, such that $\deg_{x_j}(a) > 0$ (the degree in x_j) and a unique integer $k \geq 0$ such that $p^k \leq \deg_{x_j}(a) < p^{k+1}$. Then applying the inner derivation $\mathrm{ad}\, \partial_j^{[p^k]}$ of the algebra $\mathcal{D}$ to the element a we have a *nonzero* element $\partial_j^{[p^k]} a - a \partial_j^{[p^k]} \in F_{i-p^k}$, and again induction finishes the proof of the fact that $\nu(i) \leq i$ for all $i \geq 0$. It follows that $1 \geq \mathrm{fd}(\mathcal{D}) \geq \mathrm{d}(\mathcal{D})$ (see (1)).

The $\mathcal{D}$-module P_n has dimension $\mathrm{Dim}(P_n) = n$. By Theorem 3.1,

$$\begin{aligned} 2n &= \mathrm{Dim}(\mathcal{D}) \leq \mathrm{Dim}(P_n)(\mathrm{d}(\mathcal{D}) + \max\{\mathrm{d}(\mathcal{D}), 1\}) \\ &\leq n(\mathrm{d}(\mathcal{D}) + \max\{\mathrm{d}(\mathcal{D}), 1\}), \end{aligned}$$

and so $\mathrm{d}(\mathcal{D}) \geq 1$. Then $\mathrm{d}(\mathcal{D}) = 1$, as required. $\square$

Theorem 4.3 *(***An analogue of the inequality of Bernstein***) Let M be a nonzero finitely generated $\mathcal{D}(P_n)$-module where K is a field of characteristic $p > 0$. Then $\mathrm{Dim}(M) \geq n$.*

Proof. By Theorems 3.1 and 4.2,

$$\mathrm{Dim}(M) \geq \frac{\mathrm{Dim}(\mathcal{D}(P_n))}{1+1} = \frac{2n}{n} = n. \quad \square$$

So, for any nonzero finitely generated $\mathcal{D}(P_n)$-module M: $n \leq \mathrm{Dim}(M) \leq 2n$. Any intermediate *natural* number occurs: for $n = 1$, $\mathrm{Dim}(P_1) = 1$ and $\mathrm{Dim}(\mathcal{D}(P_1)) = 2$. For arbitrary n, $\mathcal{D}(P_n) = \mathcal{D}(P_1) \otimes \cdots \otimes \mathcal{D}(P_1)$ (n times). Clearly, $\mathrm{Dim}(P_1^{\otimes s} \otimes \mathcal{D}(P_1)^{\otimes(n-s)}) = s + 2(n-s) = 2n - s$. When s runs through $0, 1, \dots n$, the number $2n - s$ runs through $n, n+1, \dots, 2n$.

We say that a function $f : \mathbb{N} \to \mathbb{N}$ has **polynomial growth** if there exists a polynomial $p(t) \in \mathbb{Q}$ such that $f(i) \leq p(i)$ for $i \gg 0$. If a function has polynomial growth so does any function which is equivalent to it. We say that a filtration $\{V_i\}$ has *polynomial growth* if the function $\dim_K V_i$ has.

Definition. A finitely generated $\mathcal{D}(P_n)$-module M is called a **holonomic** module if there is a filtration of standard type on M that has polynomial growth.

Since all filtrations of standard type are equivalent, a finitely generated $\mathcal{D}(P_n)$-module M is holonomic iff all filtrations of standard type on M has polynomial growth. It follows from the definition that the class of holonomic $\mathcal{D}(P_n)$-modules is closed under sub- and factor modules, and under finite direct sums.

5 Description of finitely presented $\mathcal{D}(P_n)$-modules, multiplicity and (Hilbert) almost polynomials

In this section, we show that in prime characteristic *finitely presented* $\mathcal{D}(P_n)$-modules behave similarly as *finitely generated* $\mathcal{D}(P_n)$-modules in characteristic zero: for *each finitely*

presented $\mathcal{D}(P_n)$-module M, the Poincare series of it is a *rational* function, though its Hilbert function is not a polynomial but an *almost polynomial* and the degree of it coincides with the dimension $\mathrm{Dim}(M)$ of M (and if $M \neq 0$ then the dimension $\mathrm{Dim}(M)$ can be *any natural number* from the interval $[n, 2n]$, this gives another proof of an analogue of the inequality of Bernstein for finitely presented $\mathcal{D}(P_n)$-modules, Theorem 5.5), and the multiplicity exits for M (Theorem 5.5). The differences are (i) in prime characteristic, finitely presented $\mathcal{D}(P_n)$-modules have transparent structure and are described by Theorem 5.5, but in characteristic zero the category of finitely generated $\mathcal{D}(P_n)$-modules is far from being well-understood, (ii) for each *natural* number d such that $n < d \leq 2n$, there exists a cyclic finitely presented $\mathcal{D}(P_n)$-module M with $\mathrm{Dim}(M) = d$ and with *arbitrary small* multiplicity, Lemma 5.6 (in characteristic zero, multiplicity is a *natural* number), though the multiplicity of *every holonomic finitely presented* $\mathcal{D}(P_n)$-module is a *natural* number (Theorem 8.7), (iii) and what is completely unexpected is that *each simple finitely presented* $\mathcal{D}(P_n)$-module is *holonomic* (Corollary 5.8), and if, in addition, the field K is *algebraically closed* that the multiplicity is *always* 1 (Corollary 6.8).

Let K be an *arbitrary* field.

Quasi and almost polynomials. A function $f : \mathbb{N} \to \mathbb{N}$ is called a **quasi-polynomial** with a *period* k if there exist k polynomials $p_s(t) \in \mathbb{Q}[t]$, $s \in \mathbb{Z}/k\mathbb{Z}$, such that

$$f(i) = p_{\bar{i}}(i) \quad \text{for all} \quad i \gg 0,$$

where $\bar{i} := i + k\mathbb{Z} \in \mathbb{Z}/k\mathbb{Z}$. We say that the quasi-polynomial f has *coefficients from a set* $S \subseteq \mathbb{Q}$ if all the polynomials p_i belong to S.

A quasi-polynomial $f = (p_0, \ldots, p_{k-1})$ is called an **almost polynomial** if all the polynomials p_i have the *same* degree $\deg(f)$ and the *same* leading coefficient $\mathrm{lc}(f)$ which are called respectively the *degree* and the *leading coefficient* of f. $e(f) := \deg(f)!\mathrm{lc}(f)$ is called the *multiplicity* of f. Then $f(i) = \frac{e(f)}{d!} i^d + \cdots$, $i \gg 0$ where $d = \deg(f)$, and the three dots mean 'smaller' terms.

A function $f : \mathbb{N} \to \mathbb{N}$ is called a **somewhat polynomial** if there are two polynomials $p, q \in \mathbb{Q}[t]$ of the *same* degree d such that $p(i) \leq f(i) \leq q(i)$ for all $i \gg 0$. Then d is called the *degree* of f.

Somewhat commutative algebras. A K-algebra R is called a *somewhat commutative* algebra if it has a *finite dimensional* filtration $R = \cup_{i \geq 0} R_i$ such that $1 \in R_0$ and the associated graded algebra $\mathrm{gr}\, R$ is an *affine commutative* algebra. Then the algebra R is a Noetherian finitely generated algebra. Let us choose homogeneous R_0-algebra generators of the R_0-algebra $\mathrm{gr}\, R := \oplus_{i \geq 0} R_i / R_{i-1}$, say $y_1, \ldots, y_s$ of graded degrees $1 \leq k_1, \ldots, k_s$ respectively (that is $y_i \in R_{k_i}/R_{k_i - 1}$). A filtration $\Gamma = \{\Gamma_i\}_{i \geq 0}$ of an R-module M is called a *good* filtration if the associated graded $\mathrm{gr}\, R$-module $\mathrm{gr}_\Gamma(M) := \oplus_{i \geq 0} \Gamma_i / \Gamma_{i-1}$ is finitely generated. An R-module M has a good filtration iff it is finitely generated, and if $\{\Gamma_i\}$ and $\{\Omega_i\}$ are two good filtrations on M, then there exists a natural number t such that $\Gamma_i \subseteq \Omega_{i+t}$ and $\Omega_i \subseteq \Gamma_{i+t}$ for all $i \geq 0$. If an R-module M is finitely generated and M_0 is a finite dimensional generating subspace of M, then the standard filtration $\{R_i M_0\}$ is good. The first two statements of the following lemma are well-known by specialists (see their proofs in [1], Theorem 3.2 and Proposition 3.3).

Lemma 5.1 *Let $R=\cup_{i\geq 0}R_i$ be a somewhat commutative algebra, $k=\mathrm{lcm}(k_1,\ldots,k_s)$, M be a finitely generated R-module of Gelfand-Kirillov dimension $d=\mathrm{GK}\,(M)$ with good filtration $\Gamma=\{\Gamma_i\}$. Then*

1. *$\dim_K(\Gamma_i)=\frac{e(M)}{d!}i^d+\cdots$ is an almost polynomial of period k with coefficients from $\frac{1}{k^d d!}\mathbb{Z}$ where $e(M)\in\frac{1}{k^d}\mathbb{N}$ is called the multiplicity of M. The multiplicity does not depend on the choice of the good filtration Γ.*

2. *The Poincare series of M, $P_M(\omega):=\sum_{i\geq 0}\dim_K(\Gamma_i)\omega^i\in\mathbb{Q}(\omega)$, is a rational function of the form $\frac{f(\omega)}{\prod_{i=1}^s(1-\omega^{k_i})}$ where $f(\omega)\in\mathbb{Q}[\omega]$. The $P_M(\omega)$ has the pole of order $d+1$ at $\omega=1$, and $e(M)=e_{P_M}$ where $e_{P_M}:=(1-\omega)^{d+1}P_M(\omega)|_{\omega=1}$ is called the multiplicity of P_M.*

3. *If the elements $y_1,\ldots,y_t$ are nilpotent then the two previous statements hold replacing the number k by $\mathrm{lcm}(k_{t+1},\ldots k_s)$.*

4. *In particular, if all non-nilpotent generators of the algebra $\mathrm{gr}\,R$ have degree 1 then $P_M(\omega)=\frac{f(\omega)}{(1-\omega)^{d+1}}$ for some polynomial $f(\omega)\in\mathbb{Q}[\omega]$ such that $e(M)=f(1)\in\mathbb{N}$, and $\dim_K(\Gamma_i)=\frac{e(M)}{d!}i^d+\cdots$ for $i\gg 0$ is a polynomial of degree d with coefficients from $\frac{1}{d!}\mathbb{Z}$.*

Proof. 3. Repeat the original proof taking into account that the algebra $R_0\langle y_1,\ldots,y_t\rangle$ is finite dimensional.

4. This statement is obvious. $\Box$

Corollary 5.2 *Let $P,Q\in\mathbb{Q}(\omega)$ be rational functions having the pole at $\omega=1$ of order n and m respectively. Let $e_P>0$ and $e_Q>0$ be the multiplicities of P and Q respectively. Then $n+m-1$ and e_Pe_Q are the order of the pole at $\omega=1$ and the multiplicity of the rational function $(1-\omega)PQ$ respectively.*

Proof. The first statement is trivial, then the multiplicity of the rational function $(1-\omega)PQ$ is equal to $(1-\omega)^{n+m-1}(1-\omega)PQ|_{\omega=1}=(1-\omega)^nP(1-\omega)^mQ|_{\omega=1}=e_Pe_Q$. $\Box$

Till the end of the section K is an *arbitrary* field of *characteristic* $p>0$.

The algebras Λ_ε. For each $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_n)\in\{\pm1\}^n$, consider the commutative subalgebra $\Lambda_\varepsilon:=\Lambda_{\varepsilon_1}\otimes\cdots\otimes\Lambda_{\varepsilon_n}$ of the ring of differential operators $\mathcal{D}(P_n)=\mathcal{D}(P_1)\otimes\cdots\otimes\mathcal{D}(P_1)$ where $\Lambda_{\varepsilon_i}:=K[x_i]$, if $\varepsilon_i=1$, and $\Lambda_{\varepsilon_i}:=K[\partial_i^*]=K[\partial_i^{[1]},\partial_i^{[2]},\ldots]$, if $\varepsilon_i=-1$. The algebra Λ_ε is a tensor product of commutative algebras Λ_{ε_i}. Each of the tensor multiples is a naturally $\mathbb{N}$-graded algebra: $K[x_i]=\oplus_{j\geq 0}Kx_i^j$ and $K[\partial_i^*]=\oplus_{j\geq 0}K\partial_i^{[j]}$. So, the algebra Λ_ε is a naturally $\mathbb{N}^n$-graded algebra with respect to the tensor product of the $\mathbb{N}$-gradings:

$$\Lambda_\varepsilon=\bigoplus_{\alpha\in\mathbb{N}^n}Kl^\alpha,\quad l^\alpha l^\beta=\binom{\alpha+\beta}{\beta}_\varepsilon l^{\alpha+\beta}\quad\text{for all }\ \alpha,\beta\in\mathbb{N}^n,$$

where $l^\alpha := l^\alpha_\varepsilon := l_1^{\alpha_1}\cdots l_n^{\alpha_n}$, $l_i^{\alpha_i} := x_i^{\alpha_i}$, if $\varepsilon_i = 1$, and $l_i^{\alpha_i} := \partial_i^{[\alpha_i]}$, if $\varepsilon_i = -1$. $\binom{\alpha+\beta}{\beta}_\varepsilon := \prod_{i=1}^n \binom{\alpha_i+\beta_i}{\beta_i}_{\varepsilon_i}$ where $\binom{i}{j}_{-1} := \binom{i}{j}$ and $\binom{i}{j}_1 := 1$. The ε-*binomials* are *translation invariant*:

$$\binom{\alpha p^k}{\beta p^k}_\varepsilon = \binom{\alpha}{\beta}_\varepsilon \quad \text{for all } \ k\geq 0 \ \text{ and } \quad \alpha,\beta\in\mathbb{N}^n. \tag{16}$$

For each $k\geq 0$, $\Lambda_\varepsilon^{[p^k]} := \bigoplus_{\alpha\in\mathbb{N}^n} Kl^{\alpha p^k}$ is a subalgebra of Λ_ε. The translation invariance of the ε-binomials implies that the K-linear map

$$\Lambda_\varepsilon \to \Lambda_\varepsilon^{[p^k]}, \ \ l^\alpha \mapsto l^{\alpha p^k}, \quad \alpha\in\mathbb{N}^n,$$

is a K-*algebra isomorphism*. There exists the descending chain of subalgebras of Λ_ε:

$$\Lambda_\varepsilon := \Lambda_\varepsilon^{[p^0]} \supset \Lambda_\varepsilon^{[p]} \supset \cdots \supset \Lambda_\varepsilon^{[p^k]} \supset \cdots, \quad \cap_{k\geq 0}\Lambda_\varepsilon^{[p^k]} = K.$$

For each $k\geq 0$, let $\Lambda_{\varepsilon,[p^k]} := \bigoplus_{\alpha<\mathbf{p}^k} Kl^\alpha = \Lambda_{\varepsilon_1,[p^k]}\otimes\cdots\otimes\Lambda_{\varepsilon_n,[p^k]}$ where $\mathbf{p}^k := (p^k,\dots,p^k)$, and $\alpha<\mathbf{p}^k$ means that $0\leq\alpha_1<p^k,\dots,0\leq\alpha_n<p^k$. $\dim_K(\Lambda_{\varepsilon,[p^k]}) = p^{nk}$. The vector space $\Lambda_{\varepsilon,[p^k]}$ is an algebra iff $\varepsilon = (-1,\dots,-1)$, and, in this case, $\Lambda_{(-1,\dots,-1),[p^k]} = \Lambda_{-1,[p^k]}\otimes\cdots\otimes\Lambda_{-1,[p^k]}$ is the tensor product of commutative finite dimensional algebras where each tensor multiple, say i'th,

$$\Lambda_{-1,p^k} = K\langle\partial_i^{[1]}\rangle\otimes K\langle\partial_i^{[p]}\rangle\otimes K\langle\partial_i^{[p^2]}\rangle\otimes\cdots\otimes K\langle\partial_i^{[p^{k-1}]}\rangle \simeq (K[t]/(t^p))^{\otimes k}$$

is the tensor product of commutative local finite dimensional algebras since $K\langle\partial_i^{[p^s]}\rangle \simeq K[t]/(t^p)$ as $(\partial_i^{[p^s]})^p = 0$, $s\geq 1$. Clearly,

$$\begin{aligned}\Lambda_{\varepsilon,[p^0]} : &= K\subset\Lambda_{\varepsilon,[p]}\subset\cdots\subset\Lambda_{\varepsilon,[p^k]}\subset\cdots, \quad \Lambda_\varepsilon = \cup_{k\geq 0}\Lambda_{\varepsilon,[p^k]},\\ \Lambda_\varepsilon &= \Lambda_{\varepsilon,[p^k]}\Lambda_\varepsilon^{[p^k]} = \Lambda_{\varepsilon,[p^k]}\otimes\Lambda_\varepsilon^{[p^k]} = \Lambda_\varepsilon^{[p^k]}\otimes\Lambda_{\varepsilon,[p^k]}, \quad k\geq 0,\end{aligned}$$

and $\Lambda_{\varepsilon,[p^k]}\Lambda_{\varepsilon,[p^l]}\subseteq\Lambda_{\varepsilon,[p^{\max\{k,l\}}]}$ for $\varepsilon = (-1,\dots,-1)$ and all $k,l\geq 0$.

The subalgebra $\Lambda := \Lambda_{-1} = K[\partial^{[1]},\partial^{[2]},\dots]$ of the algebra $\mathcal{D}(K[x])$ is *not* a finitely generated algebra (since $\Lambda = \cup_{k\geq 0}\Lambda_{[p^k]}$ is the union of its proper subalgebras), it is *not* a domain (its nil-radical $\mathfrak{n}(\Lambda)$ is equal to $\Lambda_+ := \oplus_{j\geq 1}K\partial^{[j]}$), it is *not* a Noetherian algebra as

$$\Lambda_{[p],+}\otimes\Lambda^{[p]}\subset\Lambda_{[p^2],+}\otimes\Lambda^{[p^2]}\subset\cdots\subset\Lambda_{[p^k],+}\otimes\Lambda^{[p^k]}\subset\cdots$$

is a *strictly ascending* chain of ideals of the algebra Λ where $\Lambda_{[p^k],+} := \oplus_{j=1}^{p^k-1}K\partial^{[j]}$, and

$$\Lambda/(\Lambda_{[p^k],+}\otimes\Lambda^{[p^k]}) \simeq (\Lambda/\Lambda_{[p^k],+})\otimes\Lambda^{[p^k]} \simeq K\otimes\Lambda^{[p^k]}\simeq\Lambda^{[p^k]}\simeq\Lambda, \ \ k\geq 0.$$

In spite of the fact that the algebra Λ_ε, $\varepsilon = (-1,\dots,-1)$, is 'zero dimensional' ($\Lambda_\varepsilon/\mathfrak{n}(\Lambda_\varepsilon) = \Lambda_\varepsilon/\Lambda_{\varepsilon,+} = K$), it has the rich non-trivial category of modules which in turn the ring of differential operators $\mathcal{D}(P_n)$ inherits as a subcategory (via inducing).

The algebra Λ_ε is *Noetherian* iff $\varepsilon = (1, \ldots, 1)$ (in this case, it is P_n, a finitely generated Noetherian domain). If $\varepsilon \neq (1, \ldots, 1)$ then the algebra Λ_ε is not finitely generated, not Noetherian, and not a domain.

The subalgebras $\mathcal{D}(P_n)^{[p^k]}$ and $\Lambda_{[p^k]} \otimes P_n$. For each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n$,

$$\mathcal{D}(P_n) = \Lambda_\varepsilon \otimes \Lambda_{-\varepsilon} = (\bigoplus_{\alpha \in \mathbb{N}^n} K l_\varepsilon^\alpha) \otimes (\bigoplus_{\beta \in \mathbb{N}^n} K l_{-\varepsilon}^\beta) = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} K l_\varepsilon^\alpha \otimes l_{-\varepsilon}^\beta \qquad (17)$$

as K-modules. For $k \geq 0$, the vector space $\mathcal{D}(P_n)_\varepsilon^{[p^k]} := \Lambda_\varepsilon^{[p^k]} \otimes \Lambda_{-\varepsilon}^{[p^k]} = \Lambda_{-\varepsilon}^{[p^k]} \otimes \Lambda_\varepsilon^{[p^k]}$ is a subalgebra of $\mathcal{D}(P_n)$ canonically isomorphic to the K-algebra $\mathcal{D}(P_n)$ via the K-algebra isomorphism:

$$\mathcal{D}(P_n) \to \mathcal{D}(P_n)_\varepsilon^{[p^k]}, \quad l_\varepsilon^\alpha \otimes l_{-\varepsilon}^\beta \mapsto l_\varepsilon^{\alpha p^k} \otimes l_{-\varepsilon}^{\beta p^k}, \quad \alpha, \beta \in \mathbb{N}^n.$$

This follows directly form the translation invariance of the ε-binomials and from the defining relations (15) for the K-algebra $\mathcal{D}(P_n)$ since the K-algebra $\mathcal{D}(P_n)_\varepsilon^{[p^k]}$ is generated by the elements $x_1^{p^k}, \ldots, x_n^{p^k}, \partial_1^{[ip^k]}, \ldots, \partial_n^{[ip^k]}$, $i \geq 1$, that satisfy the relations (15), these relations are defining because of the decomposition (17). It is obvious that $\mathcal{D}(P_n)_\varepsilon^{[p^k]} = \mathcal{D}(P_n)_{\varepsilon'}^{[p^k]}$ for all ε and ε' but the decompositions (17) are all distinct and they will be used later in constructing various modules. So, we drop the subscript ε. There exists a descending chain of *isomorphic* subalgebras of $\mathcal{D}(P_n)$:

$$\mathcal{D}(P_n) := \mathcal{D}(P_n)^{[p^0]} \supset \mathcal{D}(P_n)^{[p]} \supset \mathcal{D}(P_n)^{[p^2]} \supset \cdots \supset \mathcal{D}(P_n)^{[p^k]} \supset \cdots, \qquad (18)$$

and $\cap_{k \geq 0} \mathcal{D}(P_n)^{[p^k]} = K$. So, the K-linear map

$$\mathcal{D}(P_n) \to \mathcal{D}(P_n)^{[p^k]}, \;\; x_i \mapsto x_i^{p^k}, \;\; \partial_i^{[j]} \mapsto \partial_i^{[jp^k]}, \qquad (19)$$

is a K-algebra *isomorphism*.

For each $k \geq 0$, the vector space $\mathcal{D}(P_n)_{\varepsilon, [p^k]} := \Lambda_{\varepsilon, [p^k]} \otimes \Lambda_{-\varepsilon, [p^k]} = \Lambda_{-\varepsilon, [p^k]} \otimes \Lambda_{\varepsilon, [p^k]}$ has dimension p^{2nk} over K. Again, it does not depend on ε, and so we drop the subscript ε, $\mathcal{D}(P_n)_{[p^k]} := \mathcal{D}(P_n)_{\varepsilon, [p^k]} = \bigoplus_{\alpha < \mathbf{p}^k, \beta < \mathbf{p}^k} K x^\alpha \partial^{[\beta]} = \bigoplus_{\alpha < \mathbf{p}^k, \beta < \mathbf{p}^k} K \partial^{[\beta]} x^\alpha$. Clearly,

$$K =: \mathcal{D}(P_n)_{[p^0]} \subset \mathcal{D}(P_n)_{[p]} \subset \mathcal{D}(P_n)_{[p^2]} \subset \cdots, \;\; \mathcal{D}(P_n) = \cup_{k \geq 0} \mathcal{D}(P_n)_{[p^k]},$$

$$\mathcal{D}(P_n) = \mathcal{D}(P_n)_{[p^k]} \otimes \mathcal{D}(P_n)^{[p^k]} = \mathcal{D}(P_n)^{[p^k]} \otimes \mathcal{D}(P_n)_{[p^k]}, \quad k \geq 0,$$

and $\mathcal{D}(P_n)_{[p^k]} \otimes \mathcal{D}(P_n)_{[p^l]} \subseteq \mathcal{D}(P_n)_{[p^{k+l}]}$, for all $k, l \geq 0$.

The case $\varepsilon = (-1, \ldots, -1)$ and the subalgebras $\Lambda_{[p^k]} \otimes P_n$. In this case, we write $\Lambda := \Lambda_{(-1, \ldots, -1)}$, $\Lambda_{[p^k]} := \Lambda_{(-1, \ldots, -1), [p^k]}$, and $\Lambda^{[p^k]} := \Lambda_{(-1, \ldots, -1)}^{[p^k]}$. Then $\mathcal{D}(P_n) = \Lambda \otimes P_n = P_n \otimes \Lambda$ and $\mathcal{D}(P_n) = \cup_{k \geq 0} \Lambda_{[p^k]} \otimes P_n$ is a union of subalgebras:

$$P_n := \Lambda_{[p^0]} \otimes P_n \subset \Lambda_{[p]} \otimes P_n \subset \Lambda_{[p^2]} \otimes P_n \subset \cdots, \quad \Lambda_{[p^k]} \otimes P_n = P_n \otimes \Lambda_{[p^k]},$$

$$\mathcal{D}(P_n) = \Lambda^{[p^k]} \otimes (\Lambda_{[p^k]} \otimes P_n) = (\Lambda_{[p^k]} \otimes P_n) \otimes \Lambda^{[p^k]}, \quad k \geq 0.$$

For each $k \geq 0$, the algebra $\Lambda_{[p^k]} \otimes P_n$ is a free left and right P_n-module of rank p^{nk}, it is a *finitely generated Noetherian* algebra having the centre $Z_k := K[x_1^{p^k}, \ldots, x_n^{p^k}]$. *The algebra* $\Lambda_{[p^k]} \otimes P_n$ *is a free* Z_k*-module of rank* p^{2nk} since $\Lambda_{[p^k]} \otimes P_n = \Lambda_{[p^k]} \otimes (\oplus_{\alpha < \mathbf{p}^k} Kx^\alpha) \otimes Z_k$. On the algebra $\Lambda_{[p^k]} \otimes P_n$, and one can consider the *induced filtration* from the canonical filtration $F = \{F_i\}$ on the algebra $\mathcal{D}(P_n)$:

$$\mathcal{T}_k = \{\mathcal{T}_{k,i} := \Lambda_{[p^k]} \otimes P_n \cap F_i = \bigoplus_{\beta < \mathbf{p}^k, |\alpha|+|\beta| \leq i} Kx^\alpha \partial^{[\beta]} = \bigoplus_{\beta < \mathbf{p}^k, |\alpha|+|\beta| \leq i} K\partial^{[\beta]} x^\alpha\}. \tag{20}$$

The associated graded algebra $\mathrm{gr}(\Lambda_{[p^k]} \otimes P_n)$ is naturally isomorphic (as a graded algebra) to the *tensor product of the commutative algebras* $\Lambda_{[p^k]} \otimes P_n$ equipped with the tensor product of the induced filtrations (from the canonical filtration on $\mathcal{D}(P_n)$). In particular, $\mathrm{gr}(\Lambda_{[p^k]} \otimes P_n)$ is an *affine commutative* algebra with the nil-radical $\Lambda_{[p^k],+} \otimes P_n$ (where $\Lambda_{[p^k],+} := \oplus_{0 \neq \beta \in \mathbf{p}^k} K\partial^{[\beta]}$) which is a completely prime ideal (a prime ideal is a completely prime if the factor ring modulo the ideal is a domain) since

$$\mathrm{gr}(\Lambda_{[p^k]} \otimes P_n)/\Lambda_{[p^k],+} \otimes P_n \simeq (\Lambda_{[p^k]}/\Lambda_{[p^k],+}) \otimes P_n \simeq K \otimes P_n \simeq P_n.$$

The algebra $\mathrm{gr}(\Lambda_{[p^k]} \otimes P_n) = \oplus_{i \geq 0} G_i$ is positively graded (with only finitely many nonzero components) where

$$G_i = \bigoplus_{\beta < \mathbf{p}^k, |\alpha|+|\beta| = i} Kx^\alpha \partial^{[\beta]}.$$

For a ring R and a natural number $n \geq 1$, $M_n(R)$ is the ring of $n \times n$ matrices with entries from R.

Lemma 5.3 *Let K be a field of characteristic $p > 0$, and $T_k := T_{k,n} := \Lambda_{[p^k]} \otimes P_n$, $k \geq 0$. Then*

1. *The algebra T_k is a somewhat commutative algebra with respect to the finite dimensional filtration $\mathcal{T}_k = \{\mathcal{T}_{k,i}\}$ having the centre $Z_k = K[x_1^{p^k}, \ldots, x_n^{p^k}]$ and $\mathrm{GK}\,(T_{k,n}) = n$. In particular, T_k is a finitely generated Noetherian algebra, and $T_{k,n} = T_{k,1}^{\otimes n}$.*

2. *The Poincare series of T_k, $P_{T_k} = \sum_{i \geq 0} \dim_K(\mathcal{T}_{k,i})\omega^i = \frac{(1+\omega+\omega^2+\cdots+\omega^{p^k-1})^n}{(1-\omega)^{n+1}}$ and the multiplicity $e(T_k) = p^{kn}$.*

3. *The Hilbert function of T_k is, in fact, a polynomial $\dim_K(\mathcal{T}_{k,i}) = \frac{p^{kn}}{n!} i^n + \cdots$, $i \gg 0$.*

4. *Let $\mathcal{Z}_k = K(x_1^{p^k}, \ldots, x_n^{p^k})$ be the field of fractions of Z_k. Then $T_k' := \mathcal{Z}_k \otimes_{Z_k} T_k \simeq M_{p^{kn}}(\mathcal{Z}_k)$, the matrix algebra.*

5. *The algebra T_k is a prime algebra of uniform dimension p^{kn}, and the localization $\mathcal{S}^{-1}T_k$ of T_k at the set $\mathcal{S}$ of all the non-zero divisors is isomorphic to the matrix algebra $M_{p^{kn}}(\mathcal{Z}_k)$.*

6. *The algebra T_k is preserved by the involution $*$, $T_k^* = T_k$, and so the algebra T_k is self-dual.*

7. *The algebra T_k is faithfully flat over its centre.*

8. *The left and right Krull dimension of the algebra T_k is n.*

9. *The left and right global dimension of the algebra T_k is n but the global dimension of the associated graded algebra $\mathrm{gr}(T_k)$ is ∞ if $k \geq 1$.*

Proof. 1. P_n is a subalgebra of T_k, and so $n = \mathrm{GK}\,(P_n) \leq \mathrm{GK}\,(T_k)$. T_k is a finitely generated Z_k-module, and so $\mathrm{GK}\,(T_k) \leq \mathrm{GK}\,(Z_k) = n$. Therefore, $\mathrm{GK}\,(T_k) = n$.

2 and 3. These statements are obvious (see Lemma 5.1 and Corollary 5.2).

4. The $\mathcal{Z}_k$-algebra $T'_k = \bigoplus_{\alpha,\beta<\mathbf{p}^k} \mathcal{Z}_k x^\alpha \partial^{[\beta]}$ has dimension p^{2nk} over the field $\mathcal{Z}_k$. Consider the T_k-module $U := T_k/(P_n \otimes \Lambda_{[p^k],+}) \simeq P_n \otimes (\Lambda_{[p^k]}/\Lambda_{[p^k],+}) \simeq P_n \otimes K \simeq P_n \overline{1}$ where $\overline{1}$ is the canonical generator of U. The T'_k-module $U' := \mathcal{Z}_k \otimes_{Z_k} U = \bigoplus_{\alpha<\mathbf{p}^k} \mathcal{Z}_k x^\alpha \overline{1}$ is simple (use the action of $\partial^{[\beta]}$ on x^α), $\dim_{\mathcal{Z}_k}(U') = p^{nk} = \sqrt{\dim_{\mathcal{Z}_k}(T'_k)}$, and $\mathrm{End}_{T'_k}(U') \simeq \bigcap_{0<\beta<\mathbf{p}^k} \mathrm{ann}\, \partial^{[\beta]} \simeq \mathcal{Z}_k$. Therefore, $T'_k \simeq M_{p^{kn}}(\mathcal{Z}_k)$.

5. Since $\mathcal{Z}_k \backslash \{0\} \subseteq \mathcal{S}$, it follows from statement 4 that $\mathcal{S}^{-1} T_k \simeq T'_k \simeq M_{p^{kn}}(\mathcal{Z}_k)$, which implies that T_k is a prime algebra of uniform dimension p^{kn}.

6 and 7. These statements are obvious.

8. By statement 6, the left and right Krull dimension of T_k are equal. By statement 7, $\mathrm{K.dim}\,(T_k) \geq \mathrm{K.dim}\,(Z_k) = n$. The algebra T_k is a finitely generated Z_k-module, hence $\mathrm{K.dim}\,(T_k) \leq \mathrm{K.dim}\,(Z_k) = n$, and so $\mathrm{K.dim}\,(T_k) = n$.

9. Straightforward. $\square$

Since the canonical generators of the commutative $\mathbb{N}$-graded algebra $\mathrm{gr}(\Lambda_{[p^k]} \otimes P_n)$ that are not nilpotent all have graded degree 1 the next result follows from Lemma 5.1.

Lemma 5.4 *Let M be a finitely generated $\Lambda_{[p^k]} \otimes P_n$-module of Gelfand-Kirillov dimension d equipped with a standard filtration $\{M_i := \mathcal{T}_{k,i} M_0\}$ where M_0 is a finite dimensional generating space for M. Then $\dim_K(M_i) = \frac{e(M)}{d!} i^d + \cdots$, $i \gg 0$, is a polynomial of degree d with coefficients from $\frac{1}{d!}\mathbb{Z}$ where $e(M) \in \mathbb{N}$ is called the multiplicity of M. The multiplicity does not depend on the choice of a good filtration Γ. The degree d can be any natural number from the interval $[0, n]$ (see Lemma 5.3).*

Let $\mathcal{D} := \mathcal{D}(P_n)$ and $T_k := \Lambda_{[p^k]} \otimes P_n$. Consider free finitely generated (left) $\mathcal{D}$-modules $\mathcal{D}^\mu$ and $\mathcal{D}^\nu$ where $\mu, \nu \geq 1$. The set $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^\mu, \mathcal{D}^\nu)$ of all the $\mathcal{D}$-module homomorphisms from $\mathcal{D}^\mu$ to $\mathcal{D}^\nu$ can be identified with the set of all $\mu \times \nu$ matrices $M_{\mu,\nu}(\mathcal{D})$ with coefficients from $\mathcal{D}$. On this occasion, it is convenient to write homomorphisms on the *right*. Then $M_{\mu,\nu}(\mathcal{D}) = M_{\mu,\nu}(\cup_{k\geq 0} T_k) = \cup_{k \geq 0} M_{\mu,\nu}(T_k)$ is the union of matrix algebras. Let M be a *finitely presented* $\mathcal{D}$-module, that is $M = \mathrm{coker}(A)$ where $\mathcal{D}^\mu \xrightarrow{A} \mathcal{D}^\nu$, $v \mapsto vA$, $v = (v_1, \ldots, v_\mu)$, and $A \in M_{\mu,\nu}(\mathcal{D})$. Then $A \in M_{\mu,\nu}(T_k)$ for some k, and $M' := \mathrm{coker}(T_k^\mu \xrightarrow{A} T_k^\nu)$ is a finitely presented T_k-module. Applying the *exact* functor $\mathcal{D} \otimes_{T_k} -$ to the exact

sequence of T_k-modules $T_k^\mu \xrightarrow{A} T_k^\nu \to M' \to 0$ one obtains the exact sequence of $\mathcal{D}$-modules $\mathcal{D}_k^\mu \xrightarrow{A} \mathcal{D}_k^\nu \to \mathcal{D}\otimes_{T_k} M' \to 0$. Therefore,

$$M \simeq \mathcal{D}\otimes_{T_k} M', \tag{21}$$

and so *each finitely presented $\mathcal{D}$-module is isomorphic to an induced module from a finitely generated T_k-module.* The next result describes finitely presented $\mathcal{D}(P_n)$-modules and gives as a result an analogue of the inequality of Bernstein for them.

Theorem 5.5 *Let M be a nonzero finitely presented $\mathcal{D}(P_n)$-module. Then $M\simeq \mathcal{D}\otimes_{T_k} M'$ for a finitely generated T_k-module M'. Let $\{M_i'\}$ be a standard filtration for the T_k-module M' from Lemma 5.4 and $\dim_K(M_i') = \frac{e(M')}{d!}i^d+\cdots$ for $i\gg 0$ where $d=\mathrm{GK}\,(M')$. Let $\{M_i:=F_iM_0'\}$ be the filtration of standard type on the $\mathcal{D}(P_n)$-module M. Then*

1. *$\dim_K(M_i) = \frac{e(M')}{p^{kn}(n+d)!}i^{n+d}+\cdots$ is an almost polynomial of period p^k with coefficients from $\frac{1}{p^{k(n+d)}(n+d)!}\mathbb{Z}$, and $e(M) = \frac{e(M')}{p^{kn}}\in\frac{1}{p^{kn}}\mathbb{N}$.*

2. *The dimension $\mathrm{Dim}(M) = n+d\geq n$ is equal to $t-1$ where t is the order of the pole of the Poincare series $P_M(\omega) = \sum_{i\geq 0}\dim_K(M_i)\omega^i$ at the point $\omega=1$, and the multiplicity $e(M) = (1-\omega)^{\mathrm{Dim}(M)+1}P_M(\omega)|_{\omega=1}$. The dimension $\mathrm{Dim}(M)$ of M can be any natural number from the interval $[n,2n]$.*

Proof. The subalgebra $\Lambda^{[p^k]}$ of $\mathcal{D}(P_n)$ has the induced filtration $\{\Lambda_i^{[p^k]}:=\Lambda^{[p^k]}\cap F_i = \bigoplus_{p^k|\beta|\leq i}K\partial^{[p^k\beta]}\}$. Therefore,

$$P:=\sum_{i\geq 0}\dim_K(\Lambda_i^{[p^k]})\omega^i = \frac{1}{(1-\omega)(1-\omega^{p^k})^n} \quad\text{and}\quad e_P:=(1-\omega)^{n+1}P|_{\omega=1}=\frac{1}{p^{kn}}.$$

It follows from the equality $M=\Lambda^{[p^k]}\otimes M'$ that $M_i = \sum_{j+k\leq i}\Lambda_j^{[p^k]}\otimes M_k'$. Therefore, $R:=\sum_{i\geq 0}\dim_K(M_i)\omega^i = (1-\omega)PQ$ where $Q:=\sum_{i\geq 0}\dim_K(M_i')\omega^i$. By Corollary 5.2, $e(M) = e_R = e_Pe_Q = \frac{1}{p^{kn}}e(M')$ and $\mathrm{Dim}(M) = n+d\geq n$, and so $\dim_K(M_i) = \frac{e(M')}{p^{kn}(n+d)!}i^{n+d}+\cdots$, by Lemma 5.1. The rest is obvious (Lemma 5.1). $\Box$

Lemma 5.6 *For each $s=0,1,\ldots,n-1$, $\mathcal{D}(P_n) = \mathcal{D}(P_1)\otimes\mathcal{D}(P_1)^{\otimes s}\otimes\mathcal{D}(P_1)^{\otimes(n-s-1)}$. For each $k\in\mathbb{N}$, consider the cyclic finitely presented $\mathcal{D}(P_n)$-module $M(k,s):=M(k)\otimes\mathcal{D}(P_1)^{\otimes s}\otimes\Lambda_{-1}^{\otimes(n-s-1)}$ where $M(k):=\mathcal{D}(P_1)\otimes_{T_k}T_k/T_k\Lambda_{[p^k],+}$ is the $\mathcal{D}(P_1)$-module. Then $\mathrm{Dim}\,M(k,s) = n+1+s$ and $e(M(k,s)) = \frac{1}{p^k}$. So, the multiplicity of a non-holonomic finitely presented $\mathcal{D}(P_n)$-module can be arbitrary small (for each possible dimension $n,\ldots,2n$).*

Remark. By contrast, the multiplicity of each *holonomic finitely presented $\mathcal{D}(P_n)$*-module is *natural* number (Theorem 8.7).

Proof. The T_k-module $N:=T_k/T_k\Lambda_{[p^k],+} = P_1\overline{1}\simeq {}_{P_1}P_1$ has the standard filtration $\{N_i:=\mathcal{T}_{k,i}\overline{1} = \bigoplus_{j=0}^iKx_1^j\overline{1}\}$. Therefore, $\dim_K(N_i) = i+1$, and so $\mathrm{GK}\,(N)=1$ and $e(N)=$

1. By Theorem 5.5, $\mathrm{Dim}(M(k)) = 2$ and $e(M(k)) = \frac{1}{p^k}$. The $\mathcal{D}(P_1)^{\otimes s} \otimes \mathcal{D}(P_1)^{\otimes(n-s-1)}$-module $\mathcal{D}(P_1)^{\otimes s} \otimes \Lambda_{-1}^{\otimes(n-s-1)}$ has dimension $2s+n-s-1 = n+s-1$ and multiplicity 1. Using Corollary 5.2, we have $\mathrm{Dim}(M) = 2+n+s-1 = n+1+s$ and $e(M) = \frac{1}{p^k} \cdot 1 = \frac{1}{p^k}$. $\square$

Corollary 5.7 *Each short exact sequence $0 \to N \to M \to L \to 0$ of finitely presented $\mathcal{D}(P_n)$-modules is obtained from a short exact sequence $0 \to N' \to M' \to L' \to 0$ of finitely presented T_k-modules for some $k \geq 0$ by tensoring on $\mathcal{D}(P_n) \otimes_{T_k} -$.*

Proof. Let $\mathcal{D} := \mathcal{D}(P_n)$. The $\mathcal{D}$-modules N, M, and L are finitely presented, so one can fix a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{D}^{m_1} & \to & \mathcal{D}^{m_1+m_2} & \to & \mathcal{D}^{m_2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{D}^{n_1} & \to & \mathcal{D}^{n_1+n_2} & \to & \mathcal{D}^{n_2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N & \to & M & \to & L & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. One can find a (large) k such that all the matrices that correspond to the (six) maps between D^*'s have coefficients from the algebra T_k. The diagram above is obtained from the following commutative diagram (with exact rows and columns) of T_k-modules (with the same matrices = maps)

$$\begin{array}{ccccccccc}
0 & \to & T_k^{m_1} & \to & T_k^{m_1+m_2} & \to & T_k^{m_2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & T_k^{n_1} & \to & T_k^{n_1+n_2} & \to & T_k^{n_2} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N' & \to & M' & \to & L' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

by tensoring on $\mathcal{D} \otimes_{T_k} -$. $\square$

Corollary 5.8 *Each simple finitely presented $\mathcal{D}(P_n)$-module is holonomic.*

Proof. Let M be a simple finitely presented $\mathcal{D}(P_n)$-module. By Theorem 5.5, $M \simeq \mathcal{D}(P_n) \otimes_{T_k} M'$ for a finitely generated T_k-module M' which must be simple. The algebra T_k is a somewhat commutative algebra which is finitely generated module over its centre Z which is an affine algebra. Therefore, by the Quillen's Lemma ([11], 9.7.3), every element of $\mathrm{End}_{T_k}(M')$ is algebraic, this implies that each simple T_k-module is finite dimensional over the field K, and so $d = \mathrm{GK}\,(M') = 0$. By Theorem 5.5, M is a holonomic $\mathcal{D}(P_n)$-module. $\square$

Theorem 5.9 *Let M be a nonzero finitely presented $\mathcal{D}(P_n)$-module. The following statements are equivalent.*

1. *M is a holonomic $\mathcal{D}(P_n)$-module.*
2. $\mathrm{Dim}(M) = n$.
3. $\mathrm{Dim}(M) < n+1$.

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are obvious, and the implication $3 \Rightarrow 1$ follows from Theorem 5.5. $\square$

Remarks. 1. If a finitely generated $\mathcal{D}(P_n)$-module M is not finitely presented that Theorem 5.9 is not true. There exists a cyclic non-holonomic $\mathcal{D}(P_n)$-module M with $\mathrm{Dim}(M) = n$ (Proposition 9.9), and there are plenty of cyclic $\mathcal{D}(P_n)$-modules having dimension d such that $n < d < n+1$ (Theorem 9.11).

2. In characteristic zero, the multiplicity of a holonomic $\mathcal{D}(P_n)$-module is a natural number, so it can't be arbitrary small. This is the reason why each holonomic $\mathcal{D}(P_n)$-module has finite length. Though the same is true in prime characteristic (Theorem 9.6), Theorem 5.5 does *not* give a *uniform* lower bound for multiplicity of holonomic finitely presented $\mathcal{D}(P_n)$-modules, so one *cannot repeat* the arguments of the characteristic zero case even for finitely presented modules. Note that there are plenty of holonomic modules that are not finitely presented.

Theorem 5.10 *Let K be a field of characteristic $p > 0$, and $0 \to N \to M \to L \to 0$ be a short exact sequence of finitely presented $\mathcal{D}(P_n)$-modules. Then*

1. *there exist finite dimensional filtrations $\{N_i\}$, $\{M_i\}$, and $\{L_i\}$ on the modules N, M, and L respectively such that the last two are filtrations of standard type and the first one is strongly equivalent to a filtration of standard type on N and such that $\dim_K(M_i) = \dim_K(N_i) + \dim_K(L_i)$, $i \geq 0$.*
2. $\mathrm{Dim}_K(M) = \max\{\mathrm{Dim}(N), \mathrm{Dim}(L)\}$.
3. *Precisely one of the following statements is true*
   - (a) $\mathrm{Dim}(N) < \mathrm{Dim}(M) = \mathrm{Dim}(L)$ *and* $e(M) = e(L)$,
   - (b) $\mathrm{Dim}(L) < \mathrm{Dim}(M) = \mathrm{Dim}(N)$ *and* $e(M) = e(N)$,
   - (c) $\mathrm{Dim}(N) = \mathrm{Dim}(M) = \mathrm{Dim}(L)$ *and* $e(M) = e(N) + e(L)$.

Proof. 1. By Corollary 5.7, the short exact sequence $0 \to N \to M \to L \to 0$ is obtained from a short exact sequence of finitely generated T_k-modules $0 \to N' \to M' \to L' \to 0$ by tensoring on $\mathcal{D} \otimes_{T_k} -$. The algebra T_k is somewhat commutative with respect to the induced filtration $\mathcal{T}$ from the canonical filtration $F = \{F_i\}$ on the algebra $\mathcal{D} = \mathcal{D}(P_n)$. Let $\{M_i' := \mathcal{T}_{k,i} M_0\}$ be a standard filtration on the T_k-module M' and $\{L_i' := \mathcal{T}_{k,i} L_0\}$ be its image on L' which is a standard filtration on L'. It is a well-known fact

that the induced filtration $\{N'_i := N' \cap M'_i\}$ is good, and each good filtration is strongly equivalent to a standard filtration. Then $\dim_K(M'_i) = \dim_K(N'_i) + \dim_K(L'_i)$, $i \geq 0$. Since $\mathcal{D} = \Lambda^{[p^k]} \otimes T_k$ and the subalgebra $\Lambda^{[p^k]}$ of $\mathcal{D}$ has the induced filtration $\{\Lambda_i^{[p^k]} := \Lambda^{[p^k]} \cap F_i = \bigoplus_{p^k|\beta|\leq i} K\partial^{[p^k\beta]}\}$, it follows that $\{M_i := F_i M_0 = \bigoplus_{p^k|\beta|+j\leq i} \partial^{[p^k\beta]} \otimes M'_j\}$ and $\{L_i := F_i L_0 = \bigoplus_{p^k|\beta|+j\leq i} \partial^{[p^k\beta]} \otimes L'_j\}$ are filtrations of standard type on M and L respectively, and that $\{N_i := \bigoplus_{p^k|\beta|+j\leq i} \partial^{[p^k\beta]} \otimes N'_j\}$ is a finite dimensional filtration on N that is strongly equivalent to a filtration of standard type on N, and that $\dim_K(M_i) = \dim_K(N_i) + \dim_K(L_i)$, $i \geq 0$. This proves statement 1.

2 and 3. These statements follow from statement 1 and Theorem 5.5. $\square$

6 Classification of simple finitely presented $\mathcal{D}(P_n)$-modules

In this section, K is an *arbitrary* field of characteristic $p > 0$.

In this section, a classification of *simple finitely presented* $\mathcal{D}(P_n)$-modules is obtained (Theorem 6.7) which looks particularly nice for algebraically closed fields (Corollary 6.8). It will be proved that *every simple finitely presented* $\mathcal{D}(P_n)$-module M is *holonomic*, the endomorphism algebra $\mathrm{End}_{\mathcal{D}(P_n)}(M)$ is a *finite separable field* over K, and the multiplicity $e(M)$ is equal to $\dim_K(\mathrm{End}_{\mathcal{D}(P_n)}(M))$, and so it is a *natural number* (Theorem 6.7). Plenty of holonomic $\mathcal{D}(P_n)$-modules will be considered. Some of the results of this section are used as an inductive step in proving an analogue of the inequality of Bernstein in Section 9.

For an algebra A, $\widehat{A}$ denotes the set of all the isoclasses of simple A-modules, and $[M]$ denotes the isoclass of a simple A-module M.

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{\pm 1\}^n$, and let s be the number of positive coordinates of ε. The algebra Λ_ε is isomorphic to the tensor product $P_s \otimes \Lambda(t)$ of the polynomial algebra P_s and $\Lambda(t) := \Lambda_{-1}^{\otimes t}$ where $t = n - s$.

The nil-radical $\mathfrak{n}(\Lambda_\varepsilon)$ of the algebra Λ_ε is $P_s \otimes \Lambda(t)_+$ since $P_s \otimes \Lambda_+(t)$ belongs to the nil-radical of the algebra Λ_ε, and $\Lambda_\varepsilon/(P_s \otimes \Lambda_+(t)) \simeq P_s \otimes (\Lambda(t)/\Lambda(t)_+) \simeq P_s$.

Lemma 6.1 *1. Let $\Lambda = \Lambda_{(-1,\ldots,-1)}$. Then $K := \Lambda/\Lambda_+$ is the only (up to isomorphism) simple Λ-module.*

2. Let $\Lambda(t) := \Lambda_{-1}^{\otimes t}$ and $\Lambda_\varepsilon \simeq P_s \otimes \Lambda(t)$ for some $s \geq 1$ such that $s + t = n$. Then the map $\widehat{P_s} \to \widehat{\Lambda_\varepsilon}$, $[L] \mapsto [L = L \otimes \Lambda(t)/\Lambda_+(t)]$, is a bijection.

Proof. 1. Note that Λ_+ is the nil-radical of the algebra Λ and $\Lambda/\Lambda_+ = K$. Then $\widehat{\Lambda} = \widehat{\Lambda/\Lambda_+} = \widehat{K}$, and so $K = \Lambda/\Lambda_+$ is the only simple Λ-module (up to isomorphism).

2. Similarly, $\Lambda_\varepsilon/\mathfrak{n}(\Lambda_\varepsilon) \simeq P_s$. Therefore, $\widehat{\Lambda_\varepsilon} = \widehat{P_s}$, and the result follows. $\square$

Given a ring A and its element a, let $L_a(b) = ab$ and $R_a(b) = ba$. Then the maps (from A to itself), L_a, R_a, and $\mathrm{ad}\, a = L_a - R_a$ commute. Therefore, $R_a^k = (L_a - \mathrm{ad}\, a)^k =$

$\sum_{j=0}^{k}\binom{k}{j}L_a^{k-j}(-\mathrm{ad}\, a)^j$, $k\geq 0$. Applying this identity in the case where $a=x_i\in A=\mathcal{D}(P_n)$, we see that

$$\partial^{[\beta]}x_i^k=\sum_{j=0}^{\beta_i}\binom{k}{j}x_i^{k-j}\partial^{[\beta-je_i]},\ \ \beta\in\mathbb{N}^n,\ \ k\geq 0,$$

where $e_1:=(1,0,\dots,0),\dots,e_n:=(0,\dots,0,1)$ and then, for any polynomial $f\in P_n$,

$$[\partial^{[\beta]},f]=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\partial^{[\beta-e_i]}+\cdots=\sum_{i=1}^{n}\partial^{[\beta-e_i]}\frac{\partial f}{\partial x_i}+\cdots,\tag{22}$$

where the three dots denote an element of $\mathcal{D}(P_n)_{|\beta|-2}$ where $\{\mathcal{D}(P_n)_i\}_{i\geq 0}$ is the *order* filtration on $\mathcal{D}(P_n)$.

For an algebra R, R^{op} (or R^o) stands for the *opposite algebra* ($R=R^{op}$ as vector spaces but multiplication in R^{op} is given by the rule $a\cdot b=ba$).

Proposition 6.2 *Let K be a field of characteristic $p>0$, L be a simple Λ_ε-module. Then*

1. *the induced $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n)\otimes_{\Lambda_\varepsilon}L=\bigoplus_{\alpha\in\mathbb{N}^n}l^\alpha\otimes L$ is a holonomic $\mathcal{D}(P_n)$-module with $Kl^\alpha\otimes L\simeq L$ as K-modules where $\Lambda_{-\varepsilon}=\bigoplus_{\alpha\in\mathbb{N}^n}Kl^\alpha$. If, in addition, the field K is perfect then the $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n)\otimes_{\Lambda_\varepsilon}L$ is simple.*

2. *Let $F=\{F_i\}$ be the canonical filtration on $\mathcal{D}(P_n)$ and $\{F_iL\}$ be the filtration of standard type on the $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n)\otimes_{\Lambda_\varepsilon}L$. Then $\dim_K(F_iL)=\dim_K(L)\binom{i+n}{n}$ for all $i\geq 0$.*

3. *If, in addition, the field K is perfect then the endomorphism algebra $\mathrm{End}_{\mathcal{D}(P_n)}(\mathcal{D}(P_n)\otimes_{\Lambda_\varepsilon}L)$ is a finite field extension over K isomorphic to K if $\varepsilon=(-1,\dots,-1)$, and to L' if $\varepsilon\neq(-1,\dots,-1)$ where in this case $\Lambda_\varepsilon\simeq P_s\otimes\Lambda(n-s)$, $s\geq 1$, $L=L'\otimes K$ (Lemma 6.1).*

4. *The $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n)\otimes_{\Lambda_\varepsilon}L$ is finitely presented iff $\varepsilon=(1,\dots,1)$.*

Proof. It follows from the decomposition $\mathcal{D}:=\mathcal{D}(P_n)=\Lambda_{-\varepsilon}\otimes\Lambda_\varepsilon=\bigoplus_{\alpha\in\mathbb{N}^n}l^\alpha\otimes\Lambda_\varepsilon$ that $M:=\mathcal{D}\otimes_{\Lambda_\varepsilon}L\simeq\bigoplus_{\alpha\in\mathbb{N}^n}l^\alpha\otimes L$ and $Kl^\alpha\otimes L\simeq L$ as K-modules. Then it becomes obvious that, for $i\geq 0$,

$$\dim_K(F_iL)=\dim_K((\Lambda_{-\varepsilon}\cap F_i)\otimes L)=\dim_K(L)\dim_K(\Lambda_{-\varepsilon}\cap F_i)=\dim_K(L)\binom{i+n}{n}.$$

This proves statement 2 and the fact that M is a holonomic $\mathcal{D}$-module.

It follows from Lemma 6.1 that the $\mathcal{D}$-module M is finitely presented iff so is the Λ_ε-module L iff $\varepsilon=(1,\dots,1)$. This proves statement 4.

Let us prove simplicity of M in the case when the field K is perfect. If $\varepsilon=(-1,\dots,-1)$ then, by Lemma 6.1, $M=P_n$ with natural action of the ring $\mathcal{D}$ of differential operators on it, and so P_n is a simple $\mathcal{D}$-module with $\mathrm{End}_{\mathcal{D}}(P_n)=\cap_{\beta\in\mathbb{N}^n}\mathrm{ker}_{P_n}(\partial^{[\beta]})=K$.

It remains to consider the case when $\varepsilon \neq (-1, \dots, -1)$. In this case (up to order), $\Lambda_\varepsilon = P_s \otimes \Lambda(t)$ for some $s \geq 1$, $t = n - s$. By Lemma 6.1, $L = L' \otimes K$ for some finite field $L' = P_s/\mathfrak{m}$ over K where $\mathfrak{m}$ is a maximal ideal of the polynomial algebra P_s. Now, $\mathcal{D}(P_n) = \mathcal{D}(P_s) \otimes \mathcal{D}(P_t)$, $\mathcal{D}(P_s) = \Lambda(s) \otimes P_s$, and $\mathcal{D}(P_t) = P_t \otimes \Lambda(t)$. The $\mathcal{D}(P_n)$-module M is the tensor product $M_s \otimes M_t$ of the $\mathcal{D}(P_s)$-module $M_s := \mathcal{D}(P_s) \otimes_{P_s} L' \simeq \Lambda(s) \otimes L'$ and the $\mathcal{D}(P_t)$-module $M_t := \mathcal{D}(P_t) \otimes_{\Lambda(t)} \Lambda(t)/\Lambda(t)_+ = \mathcal{D}(P_t) \otimes_{\Lambda(t)} K \simeq P_t$. Moreover, $M \simeq \Lambda(s) \otimes L' \otimes P_t$. Since M_t is a simple $\mathcal{D}(P_t)$-module with $\mathrm{End}_K(M_t) = K$, to prove the fact that M is a simple $\mathcal{D}(P_n)$-module it suffices to show that M_s is a simple $\mathcal{D}(P_s)$-module. For each $i = 1, \dots, s$, the kernel of the K-algebra homomorphism $K[x_i] \to P_s \to L' = P_s/\mathfrak{m}$ is generated by an irreducible polynomial, say p_i. By the assumption, K is a *perfect* field, and so the polynomials p_i and $p_i' := \frac{dp_i}{dx_i} \neq 0$ are co-prime. Therefore, the multiplication by p_i' yields an invertible K-linear map from the field L' to itself. Let u be a nonzero element of M_s. We have to show that $\mathcal{D}(P_s)u = M_s$. We use induction on the degree d of the element $u = \sum_{|\beta|=d} \partial^{[\beta]} \otimes l_\beta + \sum_{|\beta'|<d} \partial^{[\beta']} \otimes l_{\beta'}$ where $l_\beta \in L'$ (not all are equal to zero) and $l_{\beta'} \in L'$ where $\beta, \beta' \in \mathbb{N}^s$. The first sum is called the *leading term* of the element u. The case $d = 0$ is obvious. So, let $d > 0$. There exists β in the leading term of u such that its i'th coordinate is a nonzero one and $l_\beta \neq 0$. By (22), the element

$$p_i u = -\sum_{|\beta|=d} \partial^{[\beta - e_i]} \otimes p_i' l_\beta + \cdots \neq 0,$$

has degree $< d$. Now, by induction, $\mathcal{D}(P_s)p_i u = M_s$, and so $\mathcal{D}(P_s)u = M_s$, as required. This finishes the proof of the first statement. It follows that $\cap_{i=1}^s \ker(p_i) = L'$ in M_s where $p_i : M_s \to M_s$, $v \mapsto p_i v$, which implies that $\mathrm{ann}_{M_s}(\mathfrak{m}) = L'$, but $\mathrm{ann}_{M_s}(\mathfrak{m}) \simeq \mathrm{End}_{\mathcal{D}(P_s)}(M_s)^{op}$ (here we write endomorphisms on the same side as scalars, i.e. on the left). Now,

$$\mathrm{End}_{\mathcal{D}}(M)^{op} \simeq \mathrm{ann}_M(\mathfrak{m}) \cap \mathrm{ann}_M(\Lambda(t)_+) = (K \otimes L' \otimes P_t) \cap (\Lambda(s) \otimes L' \otimes K) = K \otimes L' \otimes K \simeq L'.$$

This proves statement 3 in the case $\varepsilon \neq (-1, \dots, -1)$. The case $\varepsilon = (-1, \dots, -1)$ has been considered already. The proof of the proposition is complete. $\Box$

For any algebraic field L over K, let L^{sep} be the *maximal separable* subfield of L over K, L^{sep} is generated by all the separable subfields of L over K.

If the field K is *not* necessarily perfect then the induced module from Proposition 6.2 is not a simple module but rather *semi-simple* and its endomorphism algebra is not a field but rather a *direct product of matrix algebras* with coefficients from *separable* fields (Lemmas 6.3, 6.4, and Corollary 6.5). To prove these facts, first, we consider the simpler case when $n = 1$. A simple P_1-module L is, in fact, a field $L = K[x]/(g)$ where $P_1 = K[x]$ and $g(x) := f(x^{p^k})$ is an irreducible polynomial such that $f(t) \in K[t]$ is an irreducible *separable* polynomial ($\frac{df}{dt} \neq 0$) and $k \geq 0$. Then $L' := K[x^{p^k}]/(g) \simeq K[t]/(f(t))$ is a finite separable field extension of K, $[L' : K] = \deg_t(f(t))$. Clearly, $L' = L^{sep}$.

Lemma 6.3 *Let K, L, and g be as above.*

1. *The factor algebra* $\overline{A} := \Lambda_{[p^k]} \otimes P_1/(g)$ *of the subalgebra* $A := T_k := \Lambda_{[p^k]} \otimes P_1$ *of* $\mathcal{D}(P_1)$ *at the central element* g *is isomorphic to the matrix algebra* $M_{p^k}(L')$ *of rank* $[L : L^{sep}] = p^k$ *with coefficients from the field* $L^{sep} := L' := K[x^{p^k}]/(g)$.

2. $\mathrm{End}_{\mathcal{D}(P_1)}(\mathcal{D}(P_1) \otimes_{P_1} L)^{op} \simeq \overline{A}$.

3. *The* $\mathcal{D}(P_1)$*-module* $\mathcal{D}(P_1) \otimes_{P_1} L$ *is a semi-simple module isomorphic to a direct sum of* p^k *copies of the simple* $\mathcal{D}(P_1)$*-module* $U := \mathcal{D}(P_1) \otimes_A A/A(g, \Lambda_{[p^k],+}) = \Lambda^{[p^k]} \otimes L$*, and* $\mathrm{End}_{\mathcal{D}(P_1)}(U)^{op} \simeq L'$.

4. *The map from the set of left ideals of the algebra* $\overline{A}$ *to the set of* $\mathcal{D}(P_1)$*-submodules of the induced module* $\mathcal{D}(P_1) \otimes_{P_1} L$ *given by the rule* $\overline{V} \mapsto \mathcal{D}(P_1) \otimes_A \overline{V}$ *is a bijection with inverse* $N \mapsto N \cap \overline{A}$.

5. *The induced* $\mathcal{D}(P_1)$*-module* $\mathcal{D}(P_1) \otimes_{P_1} L$ *is simple iff the polynomial* g *is separable over* K *(i.e. when* $k = 0$*).*

Remarks. 1. This lemma will be used as an inductive step in Theorem 9.3 which is a key result behind the fact that every holonomic module has finite length (Theorem 9.6).

2. The opposite algebra appears in statement 2 simply because we write endomorphisms on the *same side* as scalars. The isomorphism in statement 2 is in fact an identity if one identifies the opposite algebra of the endomorphism algebra with the *idealizer* of the corresponding left ideal that defines the cyclic module.

Proof. Let $\mathcal{D} = \mathcal{D}(P_1)$, $P = P_1$, and $g = f(x^{p^k})$. Recall that $\mathcal{D} = \Lambda \otimes P = \Lambda^{[p^k]} \otimes \Lambda_{[p^k]} \otimes P = \Lambda^{[p^k]} \otimes A$ where A is a subalgebra of $\mathcal{D}$, and $K[x^{p^k}]$ is the centre of the algebra A. The induced $\mathcal{D}$-module

$$\mathcal{D} \otimes_P L \simeq \mathcal{D}/\mathcal{D}g \simeq \Lambda^{[p^k]} \otimes \overline{A} = \bigoplus_{i \geq 0} \partial^{[ip^k]} \otimes \overline{A}.$$

It follows from the decomposition $\overline{A} = \Lambda_{[p^k]} \otimes P/(g) = \oplus_{0 \leq i,j < p^k} \partial^{[i]} x^j L' = \oplus_{0 \leq i < p^k} \partial^{[i]} L$ that the algebra $\overline{A}$ is a simple algebra with the centre L' (use $\mathrm{ad}\, x$ and the fact that L is a field), and $\dim_{L'}(\overline{A}) = p^{2k}$. In order to prove that the algebra $\overline{A}$ is isomorphic to the matrix algebra $M_{p^k}(L')$ it suffices to find a simple $\overline{A}$-module U' such that $\dim_{L'}(U') = p^k$ and $\mathrm{End}_{\overline{A}}(U') \simeq L'$. One can easily verify that the module

$$U' := A/A(g, \Lambda_{[p^k],+}) \simeq \bigoplus_{0 \leq i < p^k} P\partial^{[i]}/(Pg \oplus (\bigoplus_{1 \leq i < p^k} P\partial^{[i]})) \simeq P/Pg \simeq L \qquad (23)$$

satisfy the two conditions above. This proves statement 1.

One can verify (using (19), (22), and separability of $f(t)$) that the $\mathcal{D}$-module $\mathcal{D} \otimes_A U' = \Lambda^{[p^k]} \otimes U'$ is a simple module. Now, the $\mathcal{D}$-module $\mathcal{D} \otimes_P L \simeq \Lambda^{[p^k]} \otimes \overline{A} \simeq \Lambda^{[p^k]} \otimes (U')^{p^k} \simeq (\Lambda^{[p^k]} \otimes U')^{p^k}$ is a direct sum of p^k copies of the simple $\mathcal{D}$-module $\mathcal{D} \otimes_A U'$. All the isomorphisms are natural. Since the set of elements of $\mathcal{D} \otimes_P L = \Lambda^{[p^k]} \otimes \overline{A}$ that are

annihilated by the left ideal $A(g, \Lambda_{[p^k],+})$ of the algebra A is equal to $\overline{A}$ and the $\mathcal{D}$-module $\mathcal{D} \otimes_P L$ is semi-simple, statements 2 and 4 follow at once. Statement 5 is obvious. $\square$

For each $i = 1, \dots, n$, let $L_i := K[x_i]/(g_i)$ be a simple $K[x_i]$-module where $g_i(x_i) = f_i(x_i^{p^{k_i}})$ is an *irreducible* polynomial such that $f_i(t) \in K[t]$ is an *irreducible separable* polynomial, $k_i \geq 0$, and $L_i' := K[x_i^{p^{k_i}}]/(g_i)$ is a finite separable field extension of K, $[L_i' : K] = \deg_t(f_i)$. Clearly, $L_i' = L_i^{sep}$.

Consider the $\mathcal{D}(P_n)$-module $M := \bigotimes_{i=1}^n M_i$ which is the tensor product of the induced $\mathcal{D}(K[x_i])$-modules $M_i := \mathcal{D}(K[x_i]) \otimes_{K[x_i]} L_i$. We keep the notation of Lemma 6.3 adding the subscript i in proper places when considering the module M_i. Clearly,

$$\begin{aligned} M &\simeq \mathcal{D}(P_n) \otimes_{P_n} P_n/(g_1, \dots, g_n) = \Lambda \otimes P_n/(g_1, \dots, g_n) = \Lambda \otimes (\bigotimes_{i=1}^n L_i) \\ &= \bigoplus_{\alpha \in \mathbb{N}^n} \partial^{[\alpha]} \otimes (\bigotimes_{i=1}^n L_i) \simeq \mathcal{D}(P_n)/\mathcal{D}(P_n)(g_1, \dots, g_n). \end{aligned}$$

$\{\mathcal{M}_i := F_i \cdot P_n/(g_1, \dots, g_n)\}$ is the filtration of standard type on the $\mathcal{D}(P_n)$-module M. Then

$$\dim_K(\mathcal{M}_i) = \prod_{j=1}^n [L_j : K] \binom{i+n}{n} = p^k \prod_{i=1}^n \deg_t(f_i(t)) \binom{i+n}{n}, \quad i \geq 0,$$

where $k := k_1 + \cdots + k_n$. So, M is a *holonomic cyclic finitely presented* $\mathcal{D}(P_n)$-module.

By Lemma 6.3, $\mathrm{End}_{\mathcal{D}(K[x_i])}(M_i)^{op} \simeq \overline{A}_i \simeq M_{p^{k_i}}(L_i')$. It follows that

$$\mathrm{End}_{\mathcal{D}(P_n)}(M)^{op} \simeq \bigotimes_{i=1}^n \overline{A}_i \simeq \bigotimes_{i=1}^n M_{p^{k_i}}(L_i') \simeq M_{p^k}(\bigotimes_{i=1}^n L_i') \simeq M_{p^k}(\prod_{\nu=1}^{\mu} \Gamma_\nu) \simeq \prod_{\nu=1}^{\mu} M_{p^k}(\Gamma_\nu).$$

The tensor product of separable fields $\otimes_{i=1}^n L_i'$ is a *semi-simple commutative* algebra, it is a direct product $\prod_{\nu=1}^{\mu} \Gamma_\nu$ of *finite separable fields* Γ_ν over K. The algebra $\overline{A} := \otimes_{i=1}^n \overline{A}_i$ is a semi-simple finite dimensional algebra. Let V_ν, $\nu = 1, \dots, \mu$, be a complete set of (pairwise non-isomorphic) simple $\overline{A}$-modules. Then $\dim_K(V_\nu) = p^k[\Gamma_\nu : K]$ and $\mathrm{End}_A(V_\nu)^{op} \simeq \Gamma_\nu$.

It follows from the equality $\mathcal{D}(P_n) = (\bigotimes_{i=1}^n \Lambda_i^{[p^{k_i}]}) \otimes A$ where $A := \otimes_{i=1}^n A_i$, $A_i := \Lambda_{i,[p^{k_i}]} \otimes K[x_i]$, that the $\mathcal{D}(P_n)$-module $M \simeq \mathcal{D}(P_n) \otimes_A \overline{A} \simeq (\bigoplus_{\nu=1}^{\mu} \mathcal{D}(P_n) \otimes_A V_\nu)^{p^k}$ is a direct sum of *simple* $\mathcal{D}(P_n)$-modules $U_\nu := \mathcal{D}(P_n) \otimes_A V_\nu$, and each of them occurs with the *same* multiplicity p^k. Summarizing, we have the following lemma which is a direct consequence of Lemma 6.3.

Lemma 6.4 *Let K be an arbitrary field of characteristic $p > 0$, the $\mathcal{D}(P_n)$-module $M = \otimes_{i=1}^n M_i$ be the tensor product of modules from Lemma 6.3. Then*

1. *The algebra $\overline{A} := \bigotimes_{i=1}^n \overline{A}_i \simeq \prod_{\nu=1}^{\mu} M_{p^k}(\Gamma_\nu)$ where $k := k_1 + \cdots + k_n$ and Γ_ν are finite separable field extensions of K.*

2. $\mathrm{End}_{\mathcal{D}(P_n)}(M)^{op} \simeq \overline{A}$.

3. *The $\mathcal{D}(P_n)$-module M is a semi-simple holonomic cyclic finitely presented module isomorphic to the direct sum $\oplus_{\nu=1}^{\mu} U_\nu^{p^k}$ where $U_\nu := \mathcal{D}(P_n)\otimes_A V_\nu$ is a simple holonomic finitely presented $\mathcal{D}(P_n)$-module, and $\mathrm{End}_{\mathcal{D}(P_n)}(U_\nu)^{op} \simeq \Gamma_\nu$ is a finite separable field extension of K.*

4. *On the simple $\mathcal{D}(P_n)$-module $U_\nu = (\bigotimes_{i=1}^n \Lambda_i^{p^{k_i}})\otimes V_\nu$ consider the filtration of standard type $\{U_{\nu,i} := F_i \cdot 1 \otimes V_\nu = \bigoplus_{i_1p^{k_1}+\cdots+i_np^{k_n}\leq i} \partial_1^{[i_1p^{k_1}]}\cdots\partial_n^{[i_np^{k_n}]}\otimes V_\nu\}$. Then*

   (a) *the Poincare series $P_{U_\nu} = \frac{\dim_K(V_\nu)}{(1-\omega)\prod_{i=1}^n(1-\omega^{p^{k_i}})} = \frac{p^k[\Gamma_\nu:K]}{(1-\omega)\prod_{i=1}^n(1-\omega^{p^{k_i}})}$ where $k := k_1+\cdots+k_n$,*

   (b) *the multiplicity $e(U_\nu) = [\Gamma_\nu : K] = \dim_K(\mathrm{End}_{\mathcal{D}(P_n)}(U_\nu))$,*

   (c) *$\dim_K(U_{\nu,i}) = \frac{e(U_\nu)}{n!}i^n+\cdots$, $i \gg 0$, is an almost polynomial with period $p^{\max\{k_1,\ldots,k_n\}}$.*

5. *The map from the set of left ideals of the algebra $\overline{A}$ to the set of $\mathcal{D}(P_n)$-submodules of M given by the rule $\overline{V} \mapsto \mathcal{D}(P_n)\otimes_A \overline{V}$ is a bijection with inverse $N \mapsto N \cap \overline{A}$.*

6. *The $\mathcal{D}(P_n)$-module M is simple iff all the polynomials $g_1,\ldots,g_n$ are separable (i.e. $k_1 = \cdots = k_n = 0$) and the tensor product of fields $\otimes_{i=1}^n L_i' = \otimes_{i=1}^n L_i$ is a field.*

Corollary 6.5 *Let K be an arbitrary field of characteristic $p>0$, L be a simple Λ_ε-module. Then the induced $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n)\otimes_{\Lambda_\varepsilon} L$ is a semi-simple holonomic $\mathcal{D}(P_n)$-module of finite length and $\mathrm{End}_{\mathcal{D}(P_n)}(\mathcal{D}(P_n)\otimes_{\Lambda_\varepsilon} L) \simeq \prod_{\nu=1}^{\mu} M_{n_\nu}(\Gamma_\nu)$ where Γ_ν are finite separable field extensions of K, $n_\mu \geq 0$, $M_0(\Gamma_\nu) := K$ (see the proof).*

Proof. We keep the notation of Lemma 6.2 and its proof. The case when $\varepsilon = (-1,\ldots,-1)$ has been considered already in the proof of Lemma 6.2 (in this case, $M = P_n$ and $\mathrm{End}_{\mathcal{D}(P_n)}(P_n) \simeq K$).

So, we may assume that $\varepsilon \neq (-1,\ldots,-1)$. In this case, $\Lambda_\varepsilon = P_s \otimes \Lambda(t)$ for some $s \geq 1$, $t = n-s$, and the $\mathcal{D}(P_n)$-module $M = M_s \otimes M_t$ (see the proof of Lemma 6.2) where the $\mathcal{D}(P_t)$-module M_t is equal to P_t and the $\mathcal{D}(P_s)$-module M_s is an epimorphic image of a $\mathcal{D}(P_s)$-module $M = \otimes_{i=1}^s M_i$ from Lemma 6.4. Since P_t is a simple $\mathcal{D}(P_t)$-module with $\mathrm{End}_{\mathcal{D}(P_t)}(P_t) = K$, every $\mathcal{D}(P_s)\otimes\mathcal{D}(P_t)$-submodule of $M_s\otimes M_t$ is equal to $N \otimes M_t$ for some $\mathcal{D}(P_s)$-submodule N of M_s. By Lemma 6.4, $M \simeq \oplus_{\nu=1}^{\mu} U_\nu^{n_\nu}\otimes M_t$ and $\mathrm{End}_{\mathcal{D}(P_n)}(M) \simeq \prod_{\nu=1}^{\mu} M_{n_\nu}(\Gamma_\nu)$ for some $n_\nu \geq 0$ such that $0 \leq n_\nu (\leq p^k)$ where $M_0(\Gamma_\nu) := K$. $\square$

Let $\mathrm{Max}(P_n)$ be be the set of all the maximal ideals of the polynomial algebra P_n. Let $\mathfrak{m} \in \mathrm{Max}(P_n)$, we are going to determine the structure of the induced $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n)\otimes_{P_n} P_n/\mathfrak{m}$ (Lemma 6.6, this lemma is central in proving Theorem 6.7). Note the $P_n/\mathfrak{m}$ is a finite field over K. For each $i = 1,\ldots,n$, there exists a unique monic irreducible polynomial $g_i \in K[x_i]$ such that $(g_i) = K[x_i]\cap\mathfrak{m}$, then $g_i(x_i) = f_i(x_i^{p^{k_i}})$ where $f_i(t) \in K[t]$ is a monic irreducible separable polynomial for some $k_i \geq 0$. Note that g_i, f_i, and k_i are uniquely determined by the ideal $\mathfrak{m}$. Let $k(\mathfrak{m}) = (k_1,\ldots,k_n)$, $g(\mathfrak{m}) = (g_1,\ldots,g_n)$, and $f(\mathfrak{m}) = (f_1,\ldots,f_n)$. Let $L_i := K[x_i]/(g_i)$, $L_i' := L_i^{sep} = K[x_i^{p^{k_i}}]/(g_i)$, $\bigotimes_{i=1}^n L_i' \simeq \prod_{\nu=1}^{\mu}\Gamma_\nu$

where Γ_ν are finite separable fields over K, let $1 = \sum_{\nu=1}^{\mu} e_\nu$ be the corresponding sum of primitive orthogonal idempotents. Let $A(\mathfrak{m}) := \bigotimes_{i=1}^n A(\mathfrak{m})_i$ where $A(\mathfrak{m})_i := \Lambda_{i,[p^{k_i}]} \otimes K[x_i]$, $\overline{A}(\mathfrak{m})_i := A(\mathfrak{m})_i / A(\mathfrak{m})_i g_i$,

$$\overline{A}(\mathfrak{m}) := \bigotimes_{i=1}^n \overline{A}(\mathfrak{m})_i \simeq M_{p^k}(\bigotimes_{i=1}^n L_i') \simeq \prod_{\nu=1}^{\mu} M_{p^k}(\Gamma_\nu), \quad \Lambda(\mathfrak{m}) := \bigotimes_{i=1}^n \Lambda_i^{[p^{k_i}]},$$

where $k := k_1 + \cdots + k_n$.

Let us consider the map $\prod_{\nu=1}^{\mu} \Gamma_\nu \simeq \bigotimes_{i=1}^n L_i' \to P_n/\mathfrak{m}$ that is the composition of the inclusion $\bigotimes_{i=1}^n L_i' \to \bigotimes_{i=1}^n L_i$ and the natural algebra epimorphism $\bigotimes_{i=1}^n L_i \to P_n/\mathfrak{m}$. Then there exists a *unique* ν such that the map $\Gamma_\nu \to P_n/\mathfrak{m}$ ($e_\nu \mapsto 1$) is a K-algebra monomorphism. We denote such a unique field Γ_ν by $\Gamma(\mathfrak{m})$. It is obvious that

$$\Gamma(\mathfrak{m}) = (P_n/\mathfrak{m})^{sep} \tag{24}$$

since $e_\mu \mapsto 0$, if $\mu \neq \nu$, $\bigotimes_{i=1}^n L_i \to P_n/\mathfrak{m}$ is an epimorphism, and the p^j'th ($j \gg 1$) power of each element of $\bigotimes_{i=1}^n L_i$ belongs to $\bigotimes_{i=1}^n L_i'$. The module

$$U(\mathfrak{m}) := \mathcal{D}(P_n) \otimes_{A(\mathfrak{m})} V(\mathfrak{m})$$

is a *simple holonomic finitely presented* $\mathcal{D}(P_n)$-module U_ν from Lemma 6.4 that corresponds to the field $\Gamma(\mathfrak{m}) = \Gamma_\nu$ where $V(\mathfrak{m}) := V_\nu$.

Lemma 6.6 *Keep the notation as above. For each maximal ideal $\mathfrak{m}$ of the polynomial algebra P_n, the induced $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n) \otimes_{P_n} P_n/\mathfrak{m}$ is isomorphic to $\frac{[P_n/\mathfrak{m}:K]}{[(P_n/\mathfrak{m})^{sep}:K]}$ copies of the simple holonomic finitely presented $\mathcal{D}(P_n)$-module $U(\mathfrak{m})$. In particular, the $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n) \otimes_{P_n} P_n/\mathfrak{m}$ is simple iff the field $P_n/\mathfrak{m}$ is separable.*

Proof. Applying $\mathcal{D}(P_n) \otimes_{A(\mathfrak{m})} -$ to the natural epimorphism of $A(\mathfrak{m})$-modules $\overline{A}(\mathfrak{m}) \to A(\mathfrak{m}) \otimes_{P_n} P_n/\mathfrak{m}$, we have the natural epimorphism of $\mathcal{D}(P_n)$-modules

$$\mathcal{D}(P_n) \otimes_{A(\mathfrak{m})} \overline{A}(\mathfrak{m}) \simeq \mathcal{D}(P_n) \otimes_{A(\mathfrak{m})} \prod_{\nu=1}^{\mu} M_{p^k}(\Gamma_\nu) \to \mathcal{D}(P_n) \otimes_{A(\mathfrak{m})} A(\mathfrak{m}) \otimes_{P_n} P_n/\mathfrak{m} \simeq \mathcal{D}(P_n) \otimes_{P_n} P_n/\mathfrak{m}.$$

Since $\Gamma_\mu \to 0$, if $\Gamma_\mu \neq \Gamma_\nu$, we have the natural epimorphism of $\mathcal{D}(P_n)$-modules

$$\mathcal{D}(P_n) \otimes_{A(\mathfrak{m})} M_{p^k}(\Gamma(\mathfrak{m})) \simeq U(\mathfrak{m})^{p^k} \to \mathcal{D}(P_n) \otimes_{P_n} P_n/\mathfrak{m}.$$

Therefore, $\mathcal{D}(P_n) \otimes_{P_n} P_n/\mathfrak{m} \simeq U(\mathfrak{m})^s$ for some $s \geq 1$. On the module $\mathcal{D}(P_n) \otimes_{P_n} P_n/\mathfrak{m}$ consider the filtration of standard type $\{F_i 1 \otimes P_n/\mathfrak{m} = \bigoplus_{|\beta| \leq i} \partial^{[\beta]} \otimes P_n/\mathfrak{m}\}$. Then

$$\dim_K(F_i 1 \otimes P_n/\mathfrak{m}) = [P_n/\mathfrak{m} : K] \binom{i+n}{n} = \frac{[P_n/\mathfrak{m} : K]}{n!} i^n + \cdots, \quad i \gg 0.$$

By Lemma 6.4, $\dim_K(U(\mathfrak{m})_i^s) = \frac{s[\Gamma(\mathfrak{m}):K]}{n!} i^n + \cdots$, $i \gg 0$. Since the multiplicity does not depend on a filtration of standard type, we must have $s[\Gamma(\mathfrak{m}) : K] = [P_n/\mathfrak{m} : K]$. This finishes the proof of the lemma (see (24)). $\Box$

Let $\widehat{\mathcal{D}(P_n)}$(fin. pres.) be the set of isoclasses of simple finitely presented $\mathcal{D}(P_n)$-modules. Theorem 6.7 *classifies* these modules and shows that *every* simple finitely presented $\mathcal{D}(P_n)$-module is *holonomic*.

Theorem 6.7 *Let K be a field of characteristic $p > 0$. Then*

1. *The map* $\mathrm{Max}(P_n) \to \widehat{\mathcal{D}(P_n)}(\text{fin. pres.})$, $\mathfrak{m} \mapsto [U(\mathfrak{m}) := \mathcal{D}(P_n) \otimes_{A(\mathfrak{m})} V(\mathfrak{m})]$, *is a bijection with inverse* $[M] \mapsto \mathrm{ass}_{P_n}(M)$ *(the set of all associated primes for the P_n-module M). In particular,* $\mathrm{ass}_{P_n}(U(\mathfrak{m})) = \{\mathfrak{m}\}$.

2. *Each simple finitely presented $\mathcal{D}(P_n)$-module M is a holonomic.*

3. *(An analogue of Quillen's Lemma).* $\mathrm{End}_{\mathcal{D}(P_n)}(U(\mathfrak{m})) \simeq (P_n/\mathfrak{m})^{sep}$.

4. *On the simple $\mathcal{D}(P_n)$-module $U(\mathfrak{m}) = \Lambda(\mathfrak{m}) \otimes V(\mathfrak{m})$ consider the filtration of standard type* $\{U(\mathfrak{m})_i := F_i 1 \otimes V(\mathfrak{m}) = \bigoplus_{i_1 p^{k_1} + \cdots + i_n p^{k_n} \leq i} \partial_1^{[i_1 p^{k_1}]} \cdots \partial_n^{[i_n p^{k_n}]} \otimes V(\mathfrak{m})\}$. *Then*

   (a) *the Poincare series* $P_{U(\mathfrak{m})} = \frac{p^k [(P_n/\mathfrak{m})^{sep} : K]}{(1-\omega) \prod_{i=1}^n (1-\omega^{p^{k_i}})}$, $k := k_1 + \cdots + k_n$,

   (b) *the multiplicity* $e(U(\mathfrak{m})) = [(P_n/\mathfrak{m})^{sep} : K] = \dim_K(\mathrm{End}_{\mathcal{D}(P_n)}(U(\mathfrak{m})))$ *is a natural number,*

   (c) $\dim_K(U(\mathfrak{m})_i) = \frac{[(P_n/\mathfrak{m})^{sep} : K])}{n!} i^n + \cdots$, $i \gg 0$, *is an almost polynomial with period* $p^{\max\{k_1, \ldots, k_n\}}$.

Remark. $\dim_K(U(\mathfrak{m})_i)$ is *not* a polynomial (for $i \gg 0$) iff $\max\{k_1, \ldots, k_n\} > 1$.

Proof. 1. Let M be a simple finitely presented $\mathcal{D}(P_n)$-module. By Corollary 5.8 and its proof, $M \simeq \mathcal{D}(P_n) \otimes_{T_k} M'$ is a holonomic $\mathcal{D}(P_n)$-module where M' is a simple finite dimensional T_k-module. Then M' is a finite dimensional P_n-module as $P_n \subseteq T_k$. Then the P_n-module M' contains a simple P_n-module isomorphic to $P_n/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of the algebra P_n. Then M is an epimorphic image of the $\mathcal{D}(P_n)$-module $N := \mathcal{D}(P_n) \otimes_{P_n} P_n/\mathfrak{m}$, and so $M \simeq U(\mathfrak{m})$, by Lemma 6.6. Note that $N = \cup_{i \geq 1} \mathrm{ann}(\mathfrak{m}^i)$, and so $\{\mathfrak{m}\} = \mathrm{ass}_{P_n}(N) = \mathrm{ass}_{P_n}(U(\mathfrak{m})^s) = \mathrm{ass}_{P_n}(U(\mathfrak{m}))$. Therefore, the map $\mathfrak{m} \mapsto U(\mathfrak{m})$ is a bijection with inverse $M \mapsto \mathrm{ass}_{P_n}(M)$. Statements 2–4 follow from statement 1 and Lemma 6.4. $\Box$

Corollary 6.8 *Let K be an algebraically closed field of characteristic $p > 0$. Then*

1. *The map* $\mathrm{Max}(P_n) = K^n \to \widehat{\mathcal{D}(P_n)}(\text{fin. pres.})$, $\mathfrak{m} \mapsto [U(\mathfrak{m}) := \mathcal{D}(P_n) \otimes_{P_n} P_n/\mathfrak{m}]$, *is a bijection with inverse* $[M] \mapsto \mathrm{ass}_{P_n}(M)$. *In particular,* $\mathrm{ass}_{P_n}(U(\mathfrak{m})) = \{\mathfrak{m}\}$.

2. $\mathrm{End}_{\mathcal{D}(P_n)}(U(\mathfrak{m})) \simeq K$.

3. *On the simple $\mathcal{D}(P_n)$-module $U(\mathfrak{m}) = \Lambda \otimes P_n/\mathfrak{m} = \Lambda\overline{1}$ consider the filtration of standard type $\{U(\mathfrak{m})_i := F_i\overline{1} = \bigoplus_{|\beta|\leq i} K\partial^{[\beta]}\overline{1}\}$. Then*

   (a) *the Poincare series $P_{U(\mathfrak{m})} = \frac{1}{(1-\omega)^{n+1}}$,*

   (b) *the multiplicity $e(U(\mathfrak{m})) = 1$,*

   (c) $\dim_K(U(\mathfrak{m})_i) = \binom{i+n}{n}$ *is a polynomial.*

7 Classification of tiny simple (non-finitely presented) $\mathcal{D}(P_n)$-modules

In this section, K is an *arbitrary* field of characteristic $p > 0$.

In this section, we complete a classification of the 'smallest' simple $\mathcal{D}(P_n)$-modules (see Theorems 7.1 and 6.7), they are called tiny modules. Theorem 6.7 describes the set of tiny *finitely presented* $\mathcal{D}(P_n)$-modules and Theorem 7.1 classifies the set of tiny *non-finitely presented* $\mathcal{D}(P_n)$-modules. They turned out to be *holonomic* with multiplicities which are natural numbers. Briefly, they have the same properties as simple finitely presented $\mathcal{D}(P_n)$-modules.

Let $\varepsilon \in \{\pm 1\}^n$. A Λ_ε-module M is called a *locally finite* if $\dim_K(\Lambda_\varepsilon m) < \infty$ for each element $m \in M$. We denote by $\mathcal{L}_\varepsilon$ the category of all $\mathcal{D}(P_n)$-modules that are locally finite as Λ_ε-modules. The category $\mathcal{L}_\varepsilon$ is a full subcategory of the category $\mathcal{D}(P_n)$-modules (it is closed under taking sub/factor modules and direct sums but not under infinite direct products).

Definition. A simple $\mathcal{D}(P_n)$-module from $\mathcal{L}_\varepsilon$ is called a *tiny* module. The name is inspired by Theorem 9.3 (which roughly speaking says that 'typically' $\dim_K(\Lambda_\varepsilon m) = \infty$).

Our aim is to describe the set $\widehat{\mathcal{D}(P_n)}(\mathcal{L}_\varepsilon)$ of all the isoclasses of simple $\mathcal{D}(P_n)$-modules that are locally finite over Λ_ε (Theorem 7.1 and Corollary 7.2).

For each $\mathfrak{m} \in \mathrm{Max}(\Lambda_\varepsilon)$, the $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n) \otimes_{\Lambda_\varepsilon} \Lambda_\varepsilon/\mathfrak{m} = \Lambda_{-\varepsilon} \otimes \Lambda_\varepsilon/\mathfrak{m} = \bigoplus_{\alpha\in\mathbb{N}^n} l^\alpha_{-\varepsilon} \otimes \Lambda_\varepsilon/\mathfrak{m}$ is a holonomic $\mathcal{D}(P_n)$-module as the filtration of standard type $\{F_i 1 \otimes \Lambda_\varepsilon/\mathfrak{m} = \bigoplus_{|\alpha|\leq i} l^\alpha_{-\varepsilon} \otimes \Lambda_\varepsilon/\mathfrak{m}\}$ on it has polynomial growth

$$\dim_K(F_i 1 \otimes \Lambda_\varepsilon/\mathfrak{m}) = \dim_K(\Lambda_\varepsilon/\mathfrak{m})\binom{i+n}{n}, \quad i \geq 0.$$

For each $j \geq 1$, $\mathrm{ass}_{\Lambda_\varepsilon}(\mathcal{D}(P_n) \otimes_{\Lambda_\varepsilon} \Lambda_\varepsilon/\mathfrak{m}^j) = \{\mathfrak{m}\}$ and $\mathcal{D}(P_n) \otimes_{\Lambda_\varepsilon} \Lambda_\varepsilon/\mathfrak{m}^j \in \mathcal{L}_\varepsilon$. If $\varepsilon \neq (1, \ldots, 1)$ then, for each $j \geq 2$, the cyclic $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n) \otimes_{\Lambda_\varepsilon} \Lambda_\varepsilon/\mathfrak{m}^j$ is not Noetherian as $\dim_K(\mathfrak{m}/\mathfrak{m}^2) = \infty$. It follows that each module $M \in \mathcal{L}_\varepsilon$ is an epimorphic image of a direct sum of induced modules of the type $\mathcal{D}(P_n) \otimes_{\Lambda_\varepsilon} \Lambda_\varepsilon/\mathfrak{m}^j$, and that

$$M = \bigoplus_{\mathfrak{m}\in\mathrm{Max}(\Lambda_\varepsilon)} M^{\mathfrak{m}}$$

is a direct sum of uniquely determined $\mathcal{D}(P_n)$-submodules $M^{\mathfrak{m}} := \cup_{i\geq 1}\mathrm{ann}_M(\mathfrak{m}^i)$ with $\mathrm{ass}_{\Lambda_\varepsilon}(M^{\mathfrak{m}}) = \{\mathfrak{m}\}$. Therefore, each *simple* module from the category $\mathcal{L}_\varepsilon$ is an epimorphic image of a module of type $\mathcal{D}(P_n)\otimes_{\Lambda_\varepsilon}\Lambda_\varepsilon/\mathfrak{m}$.

Example. For $\varepsilon = (1,\dots,1)$, i.e. $\Lambda_\varepsilon = P_n$, we have already got the description of the set $\widehat{\mathcal{D}(P_n)}(\mathcal{L}_\varepsilon) = \widehat{\mathcal{D}(P_n)}(\text{fin. pres.})$ (Theorem 6.7).

Theorem 7.1 *Let K be a field of characteristic $p>0$ and $\Lambda_\varepsilon = \Lambda(t)\otimes P_s$ where $t\geq 1$ and $s := n-t$ (i.e. $\varepsilon \neq (1,\dots,1)$). Then $\mathcal{D}(P_n) = \mathcal{D}(P_t)\otimes\mathcal{D}(P_s)$ and*

1. *The map $\mathrm{Max}(P_s)\rightarrow\widehat{\mathcal{D}(P_n)}(\mathcal{L}_\varepsilon)$, $\mathfrak{m}\mapsto\mathcal{U}(\mathfrak{m}) := P_t\otimes U(\mathfrak{m})$, is a bijection with inverse $M\mapsto\mathrm{ass}_{P_s}(M)$. In particular, $\mathrm{ass}_{P_s}(\mathcal{U}(\mathfrak{m})) = \{\mathfrak{m}\}$.*

2. *The map $\mathrm{Max}(\Lambda_\varepsilon)\rightarrow\widehat{\mathcal{D}(P_n)}(\mathcal{L}_\varepsilon)$, $\Lambda(t)_+\otimes P_s + \Lambda(t)\otimes\mathfrak{m}\mapsto\mathcal{U}(\mathfrak{m})$, is a bijection with inverse $M\mapsto\mathrm{ass}_{\Lambda_\varepsilon}(M)$. In particular, $\mathrm{ass}_{\Lambda_\varepsilon}(\mathcal{U}(\mathfrak{m})) = \{\Lambda(t)_+\otimes P_s + \Lambda(t)\otimes\mathfrak{m}\}$.*

3. *Each simple $\mathcal{D}(P_n)$-module from $\widehat{\mathcal{D}(P_n)}(\mathcal{L}_\varepsilon)$ is a holonomic but not finitely presented.*

4. *(An analogue of Quillen's Lemma). $\mathrm{End}_{\mathcal{D}(P_n)}(\mathcal{U}(\mathfrak{m}))\simeq\mathrm{End}_{\mathcal{D}(P_t)}(P_t)\otimes\mathrm{End}_{\mathcal{D}(P_s)}(U(\mathfrak{m}))\simeq K\otimes(P_s/\mathfrak{m})^{sep}\simeq(P_s/\mathfrak{m})^{sep}$.*

5. *On the simple $\mathcal{D}(P_n)$-module $\mathcal{U}(\mathfrak{m}) = P_t\otimes\Lambda(\mathfrak{m})\otimes V(\mathfrak{m})$ consider the filtration of standard type $\{\mathcal{U}(\mathfrak{m})_i := F_i 1\otimes 1\otimes V(\mathfrak{m}) = \bigoplus_{\alpha\in\mathbb{N}^t,|\alpha|+i_1p^{k_1}+\cdots+i_sp^{k_s}\leq i} x^\alpha\partial_1^{[i_1p^{k_1}]}\cdots\partial_s^{[i_sp^{k_s}]}\otimes V(\mathfrak{m})\}$. Then*

   (a) *the Poincare series $P_{\mathcal{U}(\mathfrak{m})} = (1-\omega)P_{P_t}P_{U(\mathfrak{m})} = \frac{p^k[(P_s/\mathfrak{m})^{sep}:K]}{(1-\omega)^{t+1}\prod_{i=1}^s(1-\omega^{p^{k_i}})}$ where $k = k_1+\cdots+k_s$,*

   (b) *the multiplicity $e(\mathcal{U}(\mathfrak{m})) = e(P_t)e(U(\mathfrak{m})) = [(P_s/\mathfrak{m})^{sep}:K]$ and $e(\mathcal{U}(\mathfrak{m})) = \dim_K(\mathrm{End}_{\mathcal{D}(P_n)}(U(\mathfrak{m})))$,*

   (c) *$\dim_K(\mathcal{U}(\mathfrak{m})_i) = \frac{[(P_s/\mathfrak{m})^{sep}:K])}{n!}i^n+\cdots$, $i\gg 0$, is an almost polynomial with period $p^{\max\{k_1,\dots,k_s\}}$.*

Remark. $\dim_K(\mathcal{U}(\mathfrak{m})_i)$ is *not* a polynomial (for $i\gg 0$) iff $\max\{k_1,\dots,k_s\}>1$.

Proof. Note that the map $\mathrm{Max}(P_s)\rightarrow\mathrm{Max}(\Lambda_\varepsilon)$, $\mathfrak{m}\mapsto\Lambda(t)_+\otimes P_s+\Lambda(t)\otimes\mathfrak{m}$, is a bijection. It follows that $P_t = \mathcal{D}(P_t)/\mathcal{D}(P_t)\Lambda(t)_+$ is a simple (non-finitely presented) $\mathcal{D}(P_t)$-module with $\mathrm{End}_{\mathcal{D}(P_t)}(P_t) = K$, and that any simple $\mathcal{D}(P_n)$ module M from $\widehat{\mathcal{D}(P_n)}(\mathcal{L}_\varepsilon)$ such that $\mathrm{ass}_{P_s}(M) = \{\mathfrak{m}\}$ is an epimorphic image of the $\mathcal{D}(P_n)$-module $P_t\otimes(\mathcal{D}(P_s)\otimes_{P_s}P_s/\mathfrak{m})$. Therefore, $M\simeq P_t\otimes U(\mathfrak{m})$ (Lemma 6.6). Now, the results follow from Theorem 6.7. $\Box$

Corollary 7.2 *Keep the notation from Theorem 7.1. If, in addition, the field K is algebraically closed then*

1. *The map $\mathrm{Max}(P_s) = K^s\rightarrow\widehat{\mathcal{D}(P_n)}(\mathcal{L}_\varepsilon)$, $\mathfrak{m}\mapsto\mathcal{U}(\mathfrak{m}) := P_t\otimes(\mathcal{D}(P_s)\otimes_{P_s}P_s/\mathfrak{m})$, is a bijection with inverse $M\mapsto\mathrm{ass}_{P_s}(M)$. In particular, $\mathrm{ass}_{P_s}(\mathcal{U}(\mathfrak{m})) = \{\mathfrak{m}\}$.*

2. *The map* $\mathrm{Max}(\Lambda_\varepsilon) \to \widehat{\mathcal{D}(P_n)}(\mathcal{L}_\varepsilon)$, $\Lambda(t)_+ \otimes P_s + \Lambda(t) \otimes \mathfrak{m} \mapsto \mathcal{U}(\mathfrak{m})$, *is a bijection with inverse* $M \mapsto \mathrm{ass}_{\Lambda_\varepsilon}(M)$. *In particular,* $\mathrm{ass}_{\Lambda_\varepsilon}(\mathcal{U}(\mathfrak{m})) = \{\Lambda(t)_+ \otimes P_s + \Lambda(t) \otimes \mathfrak{m}\}$.

3. $\mathrm{End}_{\mathcal{D}(P_n)}(\mathcal{U}(\mathfrak{m})) \simeq K$.

4. *On the simple* $\mathcal{D}(P_n)$*-module* $\mathcal{U}(\mathfrak{m}) = P_t \otimes \Lambda(s) \otimes P_s/\mathfrak{m} = P_t \otimes \Lambda(s)\overline{1}$, *consider the filtration of standard type* $\{\mathcal{U}(\mathfrak{m})_i := F_i\overline{1} = \bigoplus_{\alpha\in\mathbb{N}^t,\beta\in\mathbb{N}^s,|\alpha|+|\beta|\leq i} Kx^\alpha\partial^{[\beta]}\overline{1}\}$. *Then*

   (a) *the Poincare series* $P_{U(\mathfrak{m})} = \frac{1}{(1-\omega)^{n+1}}$,

   (b) *the multiplicity* $e(U(\mathfrak{m})) = 1$,

   (c) $\dim_K(U(\mathfrak{m})_i) = \binom{i+n}{n}$ *is a polynomial.*

8 Multiplicity of each finitely presented $\mathcal{D}(P_n)$-module is a natural number

In this section, K is an *arbitrary* field of characteristic $p > 0$.

We know already that the multiplicity of a *non-holonomic finitely presented* $\mathcal{D}(P_n)$-module can be *arbitrary small* (Lemma 5.6). In this section, we prove that the multiplicity of a *holonomic finitely presented* $\mathcal{D}(P_n)$-module is a *natural number* (Theorem 8.7). This result is a direct consequence of a classification of simple $T_{\mathbf{k}}$-modules (Theorem 8.5) and Theorem 5.5.

For each $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$, the subalgebra of $\mathcal{D}(P_n) = \bigotimes_{i=1}^n \mathcal{D}(K[x_i])$:

$$T_{\mathbf{k}} = T_{\mathbf{k},n} := \bigotimes_{i=1}^n (\Lambda_{i,[p^{k_i}]} \otimes K[x_i]) = \Lambda_{[\mathbf{p}^{\mathbf{k}}]} \otimes P_n = \bigoplus_{\beta<\mathbf{p}^{\mathbf{k}}} \partial^{[\beta]} \otimes P_n = \bigoplus_{\beta<\mathbf{p}^{\mathbf{k}}} P_n \otimes \partial^{[\beta]}$$

is a free left and right P_n-module of rank $p^{|\mathbf{k}|}$ where $\Lambda_{[\mathbf{p}^{\mathbf{k}}]} := \bigotimes_{i=1}^n \Lambda_{i,[p^{k_i}]}$, $|\mathbf{k}| := k_1 + \cdots + k_n$, and $\beta < \mathbf{p}^{\mathbf{k}}$ means $\beta_i < p^{k_i}$ for all i. It is a finitely generated Noetherian algebra with the centre $Z_{\mathbf{k}} := K[x_1^{p^{k_1}}, \dots, x_n^{p^{k_n}}]$. The algebra $T_{\mathbf{k}}$ is a free $Z_{\mathbf{k}}$-module of rank $p^{2|\mathbf{k}|}$ since $T_{\mathbf{k}} = \Lambda_{[\mathbf{p}^{\mathbf{k}}]} \otimes (\oplus_{\alpha<\mathbf{p}^{\mathbf{k}}} Kx^\alpha) \otimes Z_{\mathbf{k}}$. On the algebra $T_{\mathbf{k}}$ consider the induced filtration from the canonical filtration $F = \{F_i\}$ on the algebra $\mathcal{D}(P_n)$:

$$\mathcal{T}_{\mathbf{k}} = \{\mathcal{T}_{\mathbf{k},i} := T_{\mathbf{k}} \cap F_i = \bigoplus_{\beta<\mathbf{p}^{\mathbf{k}},|\alpha|+|\beta|\leq i} Kx^\alpha\partial^{[\beta]} = \bigoplus_{\beta<\mathbf{p}^{\mathbf{k}},|\alpha|+|\beta|\leq i} K\partial^{[\beta]}x^\alpha\}. \tag{25}$$

The filtration $\mathcal{T}_{\mathbf{k}}$ is the tensor product of the induced filtrations on each tensor multiple $\Lambda_{i,[p^{k_i}]} \otimes K[x_i]$ of the algebra $T_{\mathbf{k}}$. The associated graded algebra $\mathrm{gr}\, T_{\mathbf{k}} = \oplus_{i\geq 0} G_{\mathbf{k},i}$ is naturally isomorphic (as a graded algebra) to the *tensor product of the commutative algebras* $\Lambda_{[\mathbf{p}^{\mathbf{k}}]} \otimes P_n$ where

$$G_{\mathbf{k},i} := \bigoplus_{\beta<\mathbf{p}^{\mathbf{k}},|\alpha|+|\beta|=i} K\partial^{[\beta]}x^\alpha.$$

The grading on $\mathrm{gr}\, T_{\mathbf{k}}$ is the tensor product of natural gradings on the tensor multiples. The algebra $\mathrm{gr}\, T_{\mathbf{k}}$ is an *affine commutative* algebra with nil-radical $\Lambda_{[\mathbf{p^k}],+} \otimes P_n$ (where $\Lambda_{[\mathbf{p^k}],+} := \oplus_{0\neq\beta<\mathbf{p^k}} K\partial^{[\beta]}$) which is a prime ideal since

$$\mathrm{gr}\, T_{\mathbf{k}}/(\Lambda_{[\mathbf{p^k}],+} \otimes P_n) \simeq (\Lambda_{[\mathbf{p^k}]}/\Lambda_{[\mathbf{p^k}],+}) \otimes P_n \simeq K \otimes P_n \simeq P_n.$$

$T_0 := T_{(0,\dots,0)} = P_n$, $T_{\mathbf{k}} \subseteq T_{\mathbf{l}}$ iff $\mathbf{k} \leq \mathbf{l}$ (i.e. $k_1 \leq l_1, \dots, k_n \leq l_n$). $\mathcal{D}(P_n) = \cup_{\mathbf{k}\in\mathbb{N}^n} T_{\mathbf{k}}$, $T_{\mathbf{k}} T_{\mathbf{l}} \subseteq T_{\max(\mathbf{k},\mathbf{l})}$ where $\max(\mathbf{k},\mathbf{l}) := (\max(k_1,l_1),\dots,\max(k_n,l_n))$.

Lemma 8.1 *1. The algebra $T_{\mathbf{k}}$ is a somewhat commutative algebra with respect to the finite dimensional filtration $\mathcal{T}_{\mathbf{k}} = \{\mathcal{T}_{\mathbf{k},i}\}$ having the centre $Z_{\mathbf{k}} = K[x_1^{p^{k_1}}, \dots, x_n^{p^{k_n}}]$ and* $\mathrm{GK}\,(T_{\mathbf{k}}) = n$. *In particular, $T_{\mathbf{k}}$ is a finitely generated Noetherian algebra.*

2. *The Poincare series of $T_{\mathbf{k}}$, $P_{T_{\mathbf{k}}} = \sum_{i\geq 0} \dim_K(\mathcal{T}_{\mathbf{k},i})\omega^i = \frac{\prod_{i=1}^n (1+\omega+\omega^2+\cdots+\omega^{p^{k_i}-1})}{(1-\omega)^{n+1}}$ and the multiplicity $e(T_{\mathbf{k}}) = p^{|\mathbf{k}|}$.*

3. *The Hilbert function is, in fact, a polynomial $\dim_K(\mathcal{T}_{\mathbf{k},i}) = \frac{p^{|\mathbf{k}|}}{n!} i^n + \cdots$, $i \gg 0$.*

4. *Let $\mathcal{Z}_{\mathbf{k}} = K(x_1^{p^{k_1}}, \dots, x_n^{p^{k_n}})$ be the field of fractions of $Z_{\mathbf{k}}$. Then $T'_{\mathbf{k}} := \mathcal{Z}_{\mathbf{k}} \otimes_{Z_{\mathbf{k}}} T_{\mathbf{k}} \simeq M_{p^{|\mathbf{k}|}}(\mathcal{Z}_{\mathbf{k}})$, the matrix algebra.*

5. *The algebra $T_{\mathbf{k}}$ is a prime algebra of uniform dimension $p^{|\mathbf{k}|}$, and the localization $\mathcal{S}^{-1}T_{\mathbf{k}}$ of $T_{\mathbf{k}}$ at the set $\mathcal{S}$ of all the non-zero divisors is isomorphic to the matrix algebra $M_{p^{|\mathbf{k}|}}(\mathcal{Z}_{\mathbf{k}})$.*

6. *The algebra $T_{\mathbf{k}}$ is preserved by the involution $*$, $T_{\mathbf{k}}^* = T_{\mathbf{k}}$, and so the algebra $T_{\mathbf{k}}$ is self-dual.*

7. *The algebra $T_{\mathbf{k}}$ is faithfully flat over its centre.*

8. *The left and right Krull dimension of the algebra $T_{\mathbf{k}}$ is n.*

9. *The left and right global dimension of the algebra $T_{\mathbf{k}}$ is n but the global dimension of the associated graded algebra $\mathrm{gr}(T_{\mathbf{k}})$ is ∞ if $\mathbf{k} \neq 0$.*

Proof. Repeat the proof of Lemma 5.3. $\Box$

Recall that the algebra $T_{\mathbf{k}} = T_{\mathbf{k},n}$ is a somewhat commutative algebra with respect to the filtration $\mathcal{T}_{\mathbf{k}}$.

Lemma 8.2 *Let M be a finitely generated $T_{\mathbf{k}}$-module, $\mathbf{k} \leq \mathbf{l}$, and $M' = T_{\mathbf{l}} \otimes_{T_{\mathbf{k}}} M$ be a $T_{\mathbf{l}}$-module. Then*

1. $\mathrm{GK}\,({}_{T_{\mathbf{l}}}M') = \mathrm{GK}\,({}_{T_{\mathbf{k}}}M)$.

2. $e({}_{T_{\mathbf{l}}}M') = p^{|\mathbf{l}-\mathbf{k}|} e({}_{T_{\mathbf{k}}}M)$.

Proof. Let M_0 be a finite dimensional generating subspace for the $T_{\mathbf{k}}$-module $M = T_{\mathbf{k}}M_0$, $M_i := \mathcal{T}_{\mathbf{k},i}M_0$, $i \geq 0$. Then $\dim_K(M_i) = \frac{e(M)}{d!}i^d + \cdots$, $i \gg 0$ where $d = \mathrm{GK}\,(M)$. $M' = \bigoplus_{0\leq\beta<\mathbf{p}^{\mathbf{l}-\mathbf{k}}} \partial^{[\mathbf{p}^{\mathbf{k}}\beta]} \otimes M$ where $\partial^{[\mathbf{p}^{\mathbf{k}}\beta]} := \partial_1^{[p^{k_1}\beta_1]} \cdots \partial_n^{[p^{k_n}\beta_n]}$, and

$$\bigoplus_{0\leq\beta<\mathbf{p}^{\mathbf{l}-\mathbf{k}}} \partial^{[\mathbf{p}^{\mathbf{k}}\beta]} \otimes M_{i-p^{|\mathbf{l}|}} \subseteq M_i' := \mathcal{T}_{\mathbf{l},i}M_0 \subseteq \bigoplus_{0\leq\beta<\mathbf{p}^{\mathbf{l}-\mathbf{k}}} \partial^{[\mathbf{p}^{\mathbf{k}}\beta]} \otimes M_i, \quad i \gg 0.$$

Therefore,

$$\begin{aligned} \frac{p^{|\mathbf{l}-\mathbf{k}|}e(M)}{d!}i^d + \cdots &= p^{|\mathbf{l}-\mathbf{k}|}\dim_K(M_{i-p^{|\mathbf{l}|}}) \leq \dim_K(M_i') = \frac{e(M')}{d!}i^d + \cdots \\ &\leq p^{|\mathbf{l}-\mathbf{k}|}\dim_K(M_i) = \frac{p^{|\mathbf{l}-\mathbf{k}|}e(M)}{d!}i^d + \cdots, \end{aligned}$$

and so $\mathrm{GK}\,({}_{T_\mathbf{l}}M') = \mathrm{GK}\,({}_{T_\mathbf{k}}M)$ and $e({}_{T_\mathbf{l}}M') = p^{|\mathbf{l}-\mathbf{k}|}e({}_{T_\mathbf{k}}M)$. $\square$

Theorem 8.3 *Let $M' = T_{\mathbf{k}}M_0'$ be a nonzero finitely generated $T_{\mathbf{k}}$-module, $\dim_K(M_0') < \infty$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$, $k = \max(k_1, \ldots, k_n)$, and $M := \mathcal{D}(P_n) \otimes_{T_{\mathbf{k}}} M'$. Let $\{M_i' := \mathcal{T}_{\mathbf{k},i}M_0'\}$ be a standard filtration for the $T_{\mathbf{k}}$-module M' and $\dim_K(M_i') = \frac{e(M')}{d!}i^d + \cdots$ for $i \gg 0$ where $d = \mathrm{GK}\,(M')$. Let $\{M_i := F_iM_0'\}$ be the filtration of standard type on the $\mathcal{D}(P_n)$-module M. Then*

1. $\dim_K(M_i) = \frac{e(M')}{p^{|\mathbf{k}|}(n+d)!}i^{n+d} + \cdots$ *is an almost polynomial of period p^k with coefficients from $\frac{1}{p^{k(n+d)}(n+d)!}\mathbb{Z}$, and $e(M) = \frac{e(M')}{p^{|\mathbf{k}|}} \in \frac{1}{p^{|\mathbf{k}|}}\mathbb{N}$.*

2. *The dimension $\mathrm{Dim}(M) = n + d \geq n$ is equal to $t - 1$ where t is the order of the pole of the Poincare series $P_M(\omega) = \sum_{i\geq 0} \dim_K(M_i)\omega^i$ at the point $\omega = 1$, and the multiplicity $e(M) = (1-\omega)^{\mathrm{Dim}(M)+1}P_M(\omega)|_{\omega=1}$. The dimension $\mathrm{Dim}(M)$ of M can be any natural number from the interval $[n, 2n]$.*

Proof. The subalgebra $\Lambda^{[p^{\mathbf{k}}]} := \bigotimes_{i=1}^n \Lambda_i^{[p^{k_i}]}$ of $\mathcal{D}(P_n)$ has the induced filtration

$$\{\Lambda_i^{[p^{\mathbf{k}}]} := \Lambda^{[p^{\mathbf{k}}]} \cap F_i = \bigoplus_{p^{k_1}\beta_1+\cdots+p^{k_n}\beta_n\leq i} K\partial_1^{[p^{k_1}\beta_1]} \cdots \partial_n^{[p^{k_n}\beta_n]}\}.$$

Therefore,

$$P := \sum_{i\geq 0} \dim_K(\Lambda_i^{[p^{\mathbf{k}}]})\omega^i = \frac{1}{(1-\omega)\prod_{i=1}^n(1-\omega^{p^{k_i}})} \quad \text{and} \quad e_P := (1-\omega)^{n+1}P|_{\omega=1} = \frac{1}{p^{|\mathbf{k}|}}.$$

It follows from the equality $M = \Lambda^{[p^{\mathbf{k}}]} \otimes M'$ that $M_i = \sum_{s+t\leq i} \Lambda_s^{[p^{\mathbf{k}}]} \otimes M_t'$. Therefore, $R := \sum_{i\geq 0} \dim_K(M_i)\omega^i = (1-\omega)PQ$ where $Q := \sum_{i\geq 0} \dim_K(M_i')\omega^i$. By Corollary 5.2, $e(M) = e_R = e_Pe_Q = \frac{1}{p^{|\mathbf{k}|}}e(M')$ and $\mathrm{Dim}(M) = n + d \geq n$, and so $\dim_K(M_i) = \frac{e(M')}{p^{|\mathbf{k}|}(n+d)!}i^{n+d} + \cdots$, by Lemma 5.1. The rest is obvious (Lemma 5.1). $\square$

Theorem 8.4 *(A classification of simple T_k-modules where $T_k = T_{k,1}$). Let K be a field of characteristic $p > 0$ and $k \geq 0$.*

1. *The map* $\mathrm{Max}(K[x]) \to \widehat{T}_k$, $\mathfrak{m} \mapsto [T_k(\mathfrak{m})]$ *is a bijection with inverse* $[M] \mapsto \mathrm{ass}_{K[x]}(M)$ *where*

$$T_k(\mathfrak{m}) := \begin{cases} T_k \otimes_{T_{k(\mathfrak{m})}} T_{k(\mathfrak{m})}/T_{k(\mathfrak{m})}(\mathfrak{m}, \Lambda_{[p^{k(\mathfrak{m})}],+}) \simeq T_k \otimes_{T_{k(\mathfrak{m})}} K[x]/\mathfrak{m} & , k \geq k(\mathfrak{m}), \\ T_k/T_k(\mathfrak{m}, \Lambda_{[p^k],+}) \simeq K[x]/\mathfrak{m} & , k < k(\mathfrak{m}). \end{cases}$$

2. $$\dim_K T_k(\mathfrak{m}) := \begin{cases} p^{k-k(\mathfrak{m})}[K[x]/\mathfrak{m} : K] = p^k[(K[x]/\mathfrak{m})^{sep} : K] & , k \geq k(\mathfrak{m}), \\ [K[x]/\mathfrak{m} : K] = p^{k(\mathfrak{m})}[(K[x]/\mathfrak{m})^{sep} : K] & , k < k(\mathfrak{m}), \end{cases}$$

   and so

$$\frac{\dim_K T_k(\mathfrak{m})}{p^k[(K[x]/\mathfrak{m})^{sep} : K]} := \begin{cases} 1 & , k \geq k(\mathfrak{m}), \\ p^{k(\mathfrak{m})-k} & , k < k(\mathfrak{m}). \end{cases}$$

3. $$\mathrm{End}_{T_k}(T_k(\mathfrak{m})) \simeq \begin{cases} \mathrm{End}_{T_{k(\mathfrak{m})}}(T_{k(\mathfrak{m})}(\mathfrak{m})) \simeq (K[x]/\mathfrak{m})^{sep} \simeq K[x^{p^{k(\mathfrak{m})}}]/(g) & , k \geq k(\mathfrak{m}), \\ K[x^{p^k}]/(g) & , k < k(\mathfrak{m}), \end{cases}$$

   where $\mathfrak{m} = (g)$ *and* $g = f(x^{p^{k(\mathfrak{m})}})$. $\mathrm{End}_{T_k}(T_k(\mathfrak{m}))$ *is a subfield of* $K[x]/\mathfrak{m}$ *that contains* $(K[x]/\mathfrak{m})^{sep}$. $\mathrm{End}_{T_k}(T_k(\mathfrak{m})) = (K[x]/\mathfrak{m})^{sep}$ *iff* $k \geq k(\mathfrak{m})$.

4. $$\mathcal{D}(K[x]) \otimes_{T_k} T_k(\mathfrak{m}) \simeq \begin{cases} U(\mathfrak{m}) & , k \geq k(\mathfrak{m}), \\ U(\mathfrak{m})^{p^{k(\mathfrak{m})-k}} & , k < k(\mathfrak{m}), \end{cases}$$

   where $U(\mathfrak{m})$ is the simple $\mathcal{D}(K[x])$-module from Lemma 6.3.

5. *If $k \leq k(\mathfrak{m})$ then the factor algebra* $T_k/T_k\mathfrak{m} \simeq M_{p^k}(K[x^{p^k}]/(g))$ *where* $\mathfrak{m} = (g)$.

Proof. Let $\mathcal{D} = \mathcal{D}(K[x])$.

4. Let $T_k(\mathfrak{m})$ be as in the second part of statement 1. If $k \geq k(\mathfrak{m})$ then $\mathcal{D} \otimes_{T_k} T_k(\mathfrak{m}) \simeq \mathcal{D} \otimes_{T_k} T_k \otimes_{T_{k(\mathfrak{m})}} T_{k(\mathfrak{m})}/T_{k(\mathfrak{m})}(\mathfrak{m}, \Lambda_{[p^{k(\mathfrak{m})}],+}) \simeq U(\mathfrak{m})$.

If $k < k(\mathfrak{m})$ then the $\mathcal{D}$-module $M := \mathcal{D} \otimes_{T_k} T_k(\mathfrak{m})$ is an epimorphic image of the $\mathcal{D}$-module $\mathcal{D} \otimes_{K[x]} K[x]/\mathfrak{m} \simeq U(\mathfrak{m})^s$ for some $s \geq 1$ (Lemma 6.6). Therefore, $M \simeq U(\mathfrak{m})^t$ for some t. By Theorem 5.5,

$$e(M) = p^{-k}\dim_K(T_k(\mathfrak{m})) = p^{k(\mathfrak{m})-k}[(K[x]/\mathfrak{m})^{sep} : K],$$

and by Theorem 6.7, $e(U(\mathfrak{m})^t) = t[(K[x]/\mathfrak{m})^{sep} : K]$. Therefore, $t = p^{k(\mathfrak{m})-k}$.

1. If $k \geq k(\mathfrak{m})$ then the T_k-module $T_k(\mathfrak{m})$ is simple since $\mathcal{D}_{T_k}$ is faithfully flat and the induced $\mathcal{D}$-module $\mathcal{D} \otimes_{T_k} T_k(\mathfrak{m})$ is simple (by statement 4).

If $k < k(\mathfrak{m})$ then the $K[x]$-module $T_k(\mathfrak{m})$ is simple, and so the T_k-module $T_k(\mathfrak{m})$ is simple. Now, statement 1 follows from statement 4 and Theorem 6.7.

2 and 3. These statements are obvious.

5. It follows from the decomposition

$$T_k/T_kg \simeq \Lambda_{[p^k]} \otimes K[x]/(g) = \bigoplus_{0\leq i<p^k} \partial^{[i]}K[x]/(g) = \bigoplus_{0\leq i,j<p^k} \partial^{[i]}x^jK[x^{p^k}]/(g) \qquad (26)$$

that the algebra T_k/T_kg is a simple algebra with the centre $K[x^{p^k}]/(g)$ (use $\mathrm{ad}\, x$ and the fact that the $K[x]/(g)$ is a field), and $\dim_K(T_k/T_kg) = p^k[K[x]/\mathfrak{m} : K]$. By (26), the T_k/T_kg-module

$$U' := T_k/T_k(g, \Lambda_{[p^k],+}) \simeq K[x]/\mathfrak{m}$$

is simple, $\dim_K(U') = [K[x]/\mathfrak{m} : K] = p^k[K[x^{p^k}]/(g) : K]$, and $\mathrm{End}_{T_k/T_kg}(U') \simeq K[x^{p^k}]/(g)$. This implies that $T_k/T_kg \simeq M_{p^k}(K[x^{p^k}]/(g))$ (this also proves statement 1, the case $k < k(\mathfrak{m})$). $\square$

Let $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{N}^n$. We are going to classify simple $T_\mathbf{k}$-modules (Theorem 8.5). The algebra $T_\mathbf{k}$ is a somewhat commutative algebra which is a finitely generated module over its centre. By Quillen's Lemma, *every simple $T_\mathbf{k}$-module has finite dimension over K*. Given a finite dimensional T_k-module M. Then $M = \bigoplus_{\mathfrak{m}\in \mathrm{Max}(P_n)} M^\mathfrak{m}$ is a direct sum of its submodules $M^\mathfrak{m} := \cup_{i\geq 1}\mathrm{ann}_M(\mathfrak{m}^i)$. If, in addition, the $T_\mathbf{k}$-module M is simple then $M = M^\mathfrak{m}$ for a uniquely determined maximal ideal $\mathfrak{m}$ of P_n and M is an epimorphic image of the finite dimensional $T_\mathbf{k}$-module $T_\mathbf{k}/T_\mathbf{k}\mathfrak{m} \simeq T_\mathbf{k} \otimes_{P_n} P_n/\mathfrak{m} \simeq \Lambda_{[p^\mathbf{k}]} \otimes P_n/\mathfrak{m}$, $\dim_K(T_\mathbf{k} \otimes_{P_n} P_n) = p^{|\mathbf{k}|}[P_n/\mathfrak{m} : K]$.

Suppose that $\mathbf{k} \leq k(\mathfrak{m}) := (k_1', \ldots, k_n')$ (i.e. $k_1 \leq k_1', \ldots, k_n \leq k_n'$). Let $g(\mathfrak{m}) = (g_1, \ldots, g_n)$ where $g_i(x_i) = f_i(x_i^{p^{k_i'}})$. We keep the notation as in (24). Consider natural maps

$$\prod_{\nu=1}^n \Gamma_\nu \simeq \bigotimes_{i=1}^n K[x_i^{p^{k_i'}}]/(g_i) \to \bigotimes_{i=1}^n K[x_i^{p^{k_i}}]/(g_i) \xrightarrow{\phi} \bigotimes_{i=1}^n K[x_i]/(g_i) \xrightarrow{\pi} P_n/\mathfrak{m}.$$

By (24), we have the inclusions of fields:

$$(P_n/\mathfrak{m})^{sep} = \Gamma(\mathfrak{m}) \subseteq \Gamma(\mathbf{k}, \mathfrak{m}) := \mathrm{im}(\pi \circ \phi) \subseteq P_n/\mathfrak{m}. \qquad (27)$$

Consider the factor algebra (Theorem 8.4)

$$T_\mathbf{k}/T_\mathbf{k}g(\mathfrak{m}) \simeq \bigotimes_{i=1}^n T_{k_i}/T_{k_i}g_i \simeq \bigotimes_{i=1}^n M_{p^{k_i}}(K[x^{p^{k_i}}]/(g_i)) \simeq M_{p^{|\mathbf{k}|}}(\bigotimes_{i=1}^n K[x^{p^{k_i}}]/(g_i)).$$

The $T_\mathbf{k}$-module $T_\mathbf{k}/T_\mathbf{k}\mathfrak{m}$ is, in fact, a $T_\mathbf{k}/T_\mathbf{k}g(\mathfrak{m})$-module, or even, $M_{p^{|\mathbf{k}|}}(\Gamma(\mathbf{k}, \mathfrak{m}))$-module (since $e_\mu \to 0$ if $\mu \neq \nu$, see (24)). Let

$$V(\mathbf{k}, \mathfrak{m}) := \Gamma(\mathbf{k}, \mathfrak{m})^{p^{|\mathbf{k}|}} \qquad (28)$$

be the only simple module of the matrix algebra $M(\mathbf{k},\mathfrak{m}) := M_{p^{|\mathbf{k}|}}(\Gamma(\mathbf{k},\mathfrak{m}))$. Then, $\dim_K V(\mathbf{k},\mathfrak{m}) = p^{|\mathbf{k}|}[\Gamma(\mathbf{k},\mathfrak{m}) : K]$, and

$$\mathrm{End}_{M(\mathbf{k},\mathfrak{m})}(V(\mathbf{k},\mathfrak{m})) \simeq \Gamma(\mathbf{k},\mathfrak{m}). \tag{29}$$

It follows that $T_{\mathbf{k}}/T_{\mathbf{k}}\mathfrak{m} \simeq V(\mathbf{k},\mathfrak{m})^{p^\nu}$ where

$$p^\nu = \frac{\dim_K(T_{\mathbf{k}}/T_{\mathbf{k}}\mathfrak{m})}{\dim_K(V(\mathbf{k},\mathfrak{m}))} = \frac{p^{|\mathbf{k}|}[P_n/\mathfrak{m} : K]}{p^{|\mathbf{k}|}[\Gamma(\mathbf{k},\mathfrak{m}) : K]} = \frac{[P_n/\mathfrak{m} : K]}{[\Gamma(\mathbf{k},\mathfrak{m}) : K]}$$

by (27). Therefore, $V(\mathbf{k},\mathfrak{m})$ *is the only simple* $T_{\mathbf{k}}$*-module which is annihilated by a power of the maximal ideal* $\mathfrak{m}$ *(provided* $\mathbf{k} \leq k(\mathfrak{m})$*).*

For $\alpha = (\alpha_1,\dots,\alpha_n), \beta = (\beta_1,\dots,\beta_n) \in \mathbb{N}^n$, let $\min(\alpha,\beta) = (\min(\alpha_1,\beta_1),\dots,\min(\alpha_n,\beta_n))$ and $\max(\alpha,\beta) = (\max(\alpha_1,\beta_1),\dots,\max(\alpha_n,\beta_n))$.

Theorem 8.5 *(A classification of simple* $T_{\mathbf{k}}$*-modules). Let* K *be a field of characteristic* $p > 0$ *and* $\mathbf{k} = (k_1,\dots,k_n) \in \mathbb{N}^n$*.*

1. *The map* $\mathrm{Max}(P_n) \to \widehat{T}_{\mathbf{k}}$*,* $\mathfrak{m} \mapsto [T_{\mathbf{k}}(\mathfrak{m})]$ *is a bijection with inverse* $[M] \mapsto \mathrm{ass}_{P_n}(M)$ *where*
$$T_{\mathbf{k}}(\mathfrak{m}) := \begin{cases} T_{\mathbf{k}} \otimes_{T_{k(\mathfrak{m})}} V(\mathfrak{m}) & , k \geq k(\mathfrak{m}), \\ V(\mathbf{k},\mathfrak{m}) & , k \leq k(\mathfrak{m}), \\ T_{\mathbf{k}} \otimes_{T_{\min(\mathbf{k},k(\mathfrak{m}))}} V(\min(\mathbf{k},k(\mathfrak{m})),\mathfrak{m}) & , \textit{otherwise}. \end{cases}$$

2. $$\dim_K T_{\mathbf{k}}(\mathfrak{m}) := \begin{cases} p^{|\mathbf{k}|}[(P_n/\mathfrak{m})^{sep} : K] & , k \geq k(\mathfrak{m}), \\ p^{|\mathbf{k}|}[\Gamma(\mathbf{k},\mathfrak{m}) : K] & , k \leq k(\mathfrak{m}), \\ p^{|\mathbf{k}|}[\Gamma(\min(\mathbf{k},k(\mathfrak{m})),\mathfrak{m}) : K] & , \textit{otherwise}, \end{cases}$$
$$r(\mathbf{k},\mathfrak{m}) := \frac{\dim_K T_{\mathbf{k}}(\mathfrak{m})}{p^{|\mathbf{k}|}[(P_n/\mathfrak{m})^{sep} : K]} := \begin{cases} 1 & , k \geq k(\mathfrak{m}), \\ [\Gamma(\mathbf{k},\mathfrak{m}) : (P_n/\mathfrak{m})^{sep}] & , k \leq k(\mathfrak{m}), \\ [\Gamma(\min(\mathbf{k},k(\mathfrak{m})),\mathfrak{m}) : (P_n/\mathfrak{m})^{sep}] & , \textit{otherwise}, \end{cases}$$
and $r(\mathbf{k},\mathfrak{m}) = p^s$ *for some* $s = s(\mathbf{k},\mathfrak{m}) \in \mathbb{N}$*.*

3. $$\mathrm{End}_{T_{\mathbf{k}}}(T_{\mathbf{k}}(\mathfrak{m})) \simeq \begin{cases} (P_n/\mathfrak{m})^{sep} & , k \geq k(\mathfrak{m}), \\ \Gamma(\mathbf{k},\mathfrak{m}) & , k \leq k(\mathfrak{m}), \\ \Gamma(\min(\mathbf{k},k(\mathfrak{m})),\mathfrak{m}) & , \textit{otherwise}. \end{cases}$$
$\mathrm{End}_{T_{\mathbf{k}}}(T_{\mathbf{k}}(\mathfrak{m}))$ *is a subfield of* $P_n/\mathfrak{m}$ *that contains* $(P_n/\mathfrak{m})^{sep}$*.*

4. $\dim_K T_{\mathbf{k}}(\mathfrak{m}) = p^{|\mathbf{k}|}\dim_K \mathrm{End}_{T_{\mathbf{k}}}(T_{\mathbf{k}}(\mathfrak{m}))$*.*

5. $\mathcal{D}(P_n) \otimes_{T_{\mathbf{k}}} T_{\mathbf{k}}(\mathfrak{m})) \simeq U(\mathfrak{m})^{r(\mathbf{k},\mathfrak{m})}$*.*

Proof. 1. Let $\mathfrak{m} \in \mathrm{Max}(P_n)$. If $\mathbf{k} \geq k(\mathfrak{m})$ then the $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n) \otimes_{T_{k(\mathfrak{m})}} V(\mathfrak{m}) \simeq \mathcal{D}(P_n) \otimes_{T_{\mathbf{k}}} (T_{\mathbf{k}} \otimes_{T_{k(\mathfrak{m})}} V(\mathfrak{m}))$ is simple (Theorem 6.7). Therefore, $T_{\mathbf{k}} \otimes_{T_{k(\mathfrak{m})}} V(\mathfrak{m})$ must be a simple $T_{\mathbf{k}}$-module.

If $k \leq k(\mathfrak{m})$ then $V(\mathbf{k}, \mathfrak{m})$ is a simple $T_{\mathbf{k}}$-module.

In the remaining case, one can prove that any nonzero $T_{\mathbf{k}}$-submodule of $M := T_{\mathbf{k}} \otimes_{T_{\mathbf{l}}} V(\mathbf{l}, \mathfrak{m})$, $\mathbf{l} := \min(\mathbf{k}, k(\mathfrak{m}))$, has a nonzero intersection with the simple $T_{\mathbf{l}}$-submodule $V(\mathbf{l}, \mathfrak{m})$ of M. Therefore, M is a simple $T_{\mathbf{k}}$-module. The rest of statement 1 is obvious (see Theorem 6.7 and the arguments preceding Theorem 8.5).

2. If $\mathbf{k} \geq k(\mathfrak{m})$ then

$$\dim_K(T_{\mathbf{k}}(\mathfrak{m})) = p^{|\mathbf{k}-k(\mathfrak{m})|}\dim_K V(\mathfrak{m}) = p^{|\mathbf{k}-k(\mathfrak{m})|}p^{|k(\mathfrak{m})|}[\Gamma(\mathfrak{m}) : K] = p^{|\mathbf{k}|}[(P_n/\mathfrak{m})^{sep} : K].$$

If $\mathbf{k} \leq k(\mathfrak{m})$ then the result follows from (28). In the third case, let $l = \min(\mathbf{k}, k(\mathfrak{m}))$. Then

$$\dim_K(T_{\mathbf{k}}(\mathfrak{m})) = p^{|\mathbf{k}-l|}\dim_K V(l, \mathfrak{m}) = p^{|\mathbf{k}-l|}p^{|l|}[\Gamma(l, \mathfrak{m}) : K] = p^{|\mathbf{k}|}[\Gamma(l, \mathfrak{m}) : K].$$

The rest of statement 2 is obvious.

3. Evident.

4. This follows from statement 2.

5. By Lemma 6.6, the $\mathcal{D}(P_n)$-module $N := \mathcal{D}(P_n) \otimes_{T_{\mathbf{k}}} T_{\mathbf{k}}(\mathfrak{m})$ is isomorphic to $U(\mathfrak{m})^r$ for some $r \in \mathbb{N}$. By Theorem 8.3, the multiplicity of the $\mathcal{D}(P_n)$-module N is equal to $e(N) = p^{-|\mathbf{k}|}\dim_K T_{\mathbf{k}}(\mathfrak{m})$. By Theorem 6.7, $e(U(\mathfrak{m})^r) = r[(P_n/\mathfrak{m})^{sep} : K]$, hence

$$r = \frac{\dim_K T_{\mathbf{k}}(\mathfrak{m})}{p^{|\mathbf{k}|}[(P_n/\mathfrak{m})^{sep} : K]} = r(\mathbf{k}, \mathfrak{m}). \quad \square$$

Corollary 8.6 $p^{|\mathbf{k}|} | \dim_K(M)$ *for all finite dimensional* $T_{\mathbf{k}}$*-modules* M.

Theorem 8.7 *Let M be a nonzero holonomic finitely presented $\mathcal{D}(P_n)$-module. Then its multiplicity is a natural number.*

Proof. This follows directly from Corollary 8.6, (21), and Theorem 8.3. $\square$

9 Holonomic sets of subalgebras with multiplicity, every holonomic $\mathcal{D}(P_n)$-module has finite length

In this section, K is an *arbitrary* field of characteristic $p > 0$ if it is not stated otherwise.

In this section, the concept of *holonomic set of subalgebras with multiplicity* is introduced which is a crucial one in the proof of the analogue of the inequality of Bernstein for the algebra $\mathcal{D}(P_n)$ (Theorem 9.4) and in the proof of the fact that each holonomic $\mathcal{D}(P_n)$-module has finite length and the length does not exceed the multiplicity (Theorem 9.6). It is proved that $n \leq \mathrm{Dim}(L) \leq 2n$ for each nonzero finitely generated $\mathcal{D}(P_n)$-module

L, and, for *each real* number $d \in [n, 2n]$, there exists a cyclic $\mathcal{D}(P_n)$-module M with $\mathrm{Dim}(M) = d$ (Theorem 9.11), and there exists a cyclic non-holonomic $\mathcal{D}(P_n)$-module N with $\mathrm{Dim}(N) = n$ (Proposition 9.9).

Holonomic sets of subalgebras. Let A be an algebra over an arbitrary field K with a *finite dimensional* filtration $\mathcal{A} = \{A_i\}_{i\geq 0}$ such that $\mathrm{Dim}(A) := \gamma(\dim_K A_i) < \infty$. Any subalgebra B of the algebra A has the induced finite dimensional filtration $\mathcal{B} = \{B_i := B \cap A_i\}$ and $\mathrm{Dim}(B) := \gamma(\dim_K B_i) \leq \mathrm{Dim}(A) < \infty$.

Definition. A set $\mathcal{C} = \{C_\nu\}_{\nu\in\mathcal{N}}$ of subalgebras of the algebra A is called a **sub-holonomic** set if there exists a real positive number $h_\mathcal{C}$ such that for each nonzero A-module M there exists $\nu \in \mathcal{N}$ and a finitely generated C_ν-submodule M_ν of M such that $\mathrm{Dim}({}_{C_\nu}M_\nu) \geq h_\mathcal{C}$ or, equivalently, there exists a nonzero finite dimensional vector subspace V of M such that $\gamma(\dim_K(C_{\nu,i}V)) \geq h_\mathcal{C}$ for some ν where $\{C_{\nu,i} := C_\nu \cap A_i\}$ is the induced filtration on the algebra C_ν.

Surprisingly, the following simple observation yields an idea of another proof of the inequality of Bernstein for the ring of differential operators in positive characteristic, and, more importantly, it produces an analogue of multiplicity.

Lemma 9.1 *Let A, $\mathcal{C} = \{C_\nu\}_{\nu\in\mathcal{N}}$, and $h_\mathcal{C}$ be as above. Then $\mathrm{Dim}(M) \geq h_\mathcal{C}$ for all nonzero finitely generated A-modules M.*

Proof. For a nonzero finitely generated A-module M, we have $\gamma(\dim_K C_{\nu,i}V) \geq h_\mathcal{C}$ for some nonzero finite dimensional K-subspace V of M. Let M_0 be a finite dimensional generating subspace for the A-module M that contains V. Then

$$\mathrm{Dim}(M) = \gamma(\dim_K A_i M_0) \geq \gamma(\dim_K C_{\nu,i}V) \geq h_\mathcal{C}. \quad \square$$

Definition. A set $\mathcal{C} = \{C_\nu\}_{\nu\in\mathcal{N}}$ of subalgebras of the algebra A is called a **sub-holonomic** set of **degree** n and with **leading coefficient** l where n and l are positive real numbers if for each nonzero A-module M there exists a nonzero finite dimensional K-vector subspace $V \subseteq M$ and an algebra C_ν such that $\dim_K(C_{\nu,i}V) \geq li^n + \cdots$ (where the three dots mean a function which is negligible comparing to i^n, i.e. $o(i^n)$). If n is a *natural* number then $e := n!l$ is called the **multiplicity** for $\mathcal{C}$. If, in addition, $n = h_A$ then the set $\mathcal{C}$ is called a **holonomic** set of subalgebras with **leading coefficient** l (or **multiplicity** e) for the algebra A where $h_A := \inf\{\mathrm{Dim}(M) \mid M \text{ is a nonzero finitely generated } A\text{-module}\}$ is the *holonomic number* for the algebra A with respect to the filtration $\mathcal{A}$.

Theorem 9.2 *If there exists a holonomic set $\mathcal{C} = \{C_\nu\}_{\nu\in\mathcal{N}}$ of subalgebras with the leading coefficient $l_\mathcal{C}$ for the algebra A then every holonomic A-module has finite length. Moreover, if $\{M_i\}$ is a filtration of standard type on a holonomic A-module M then the length of the A-module M is $\leq \frac{l(M)}{l_\mathcal{C}}$ where $l_\mathcal{C}$ is the leading coefficients for $\mathcal{C}$, $\dim_K(M_i) \leq l(M)i^n + \cdots$, $i \gg 0$, and the three dots mean $o(i^n)$.*

Proof. It suffices to prove the last statement. Suppose to the contrary that there exists a holonomic A-module M of length $> \frac{l(M)}{l_\mathcal{C}}$, we seek a contradiction. Then one can choose

a strictly ascending chain of submodules in M: $0 = M'_0 \subset M'_1 \subset \cdots \subset M'_t \subseteq M$ with $t > \frac{l(M)}{l_{\mathcal{C}}}$. For each factor module M'_j/M'_{j-1}, fix a nonzero finite dimensional subspace $\overline{V}_j \subseteq M'_j/M'_{j-1}$ such that $\dim_K(C_{\nu(j),i}\overline{V}_j) \geq l_{\mathcal{C}} i^n + \cdots$, $i \gg 0$, for some $\nu(j)$. Let V_j be a finite dimensional subspace of M'_j such that $\overline{V}_j = V_j + M'_{j-1}$. Fix $s \geq 1$ such that $V_1 + \cdots + V_t \subseteq M_s$. Then for $i \gg 0$,

$$\begin{aligned} t l_{\mathcal{C}} i^n + \cdots &\leq \sum_{j=1}^{t} \dim(C_{\nu(j),i}\overline{V}_j) \leq \dim(\sum_{j=1}^{t} C_{\nu(j),i} V_j) \leq \dim M_{i+s} \\ &\leq l(M)(i+s)^n + \cdots = l(M) i^n + \cdots , \end{aligned}$$

and so $t l_{\mathcal{C}} \leq l(M)$, a contradiction. $\square$

Definition. We say that a subalgebra of $\mathcal{D}(P_n)$ is of type $P_s \otimes \Lambda(n-s)$ (resp. of type $P_s \otimes \bigotimes_{i=1}^{n-s} \Lambda_{-1}^{[p^{k_i}]}$) if after changing, if necessary, the order of the tensor multiples in $\mathcal{D}(P_n) = \mathcal{D}(P_1) \otimes \cdots \otimes \mathcal{D}(P_1)$ the algebra is equal to $P_1^{\otimes s} \otimes \Lambda_{-1}^{\otimes(n-s)}$ (resp. $P_s \otimes \bigotimes_{i=1}^{n-s} \Lambda_{-1}^{[p^{k_i}]}$)).

For $\varepsilon \in \{\pm 1\}^n$, $|\varepsilon|$ denotes the number of *negative* coordinates (eg, $|(-1, \ldots, -1)| = n$ and $|(1, \ldots, 1)| = 0$).

Theorem 9.3 *Let K be an arbitrary field of characteristic $p > 0$. For any nonzero $\mathcal{D}(P_n)$-module M there exists a subalgebra Λ of the type $P_s \otimes \bigotimes_{i=1}^{n-s} \Lambda_{-1}^{[p^{k_i}]}$ of $\mathcal{D}(P_n)$ for some $k_i \geq 0$ and a finite dimensional K-subspace V of M such that $\dim_K(V) \geq p^{k_1 + \cdots + k_{n-s}}$ and the natural map $\Lambda \otimes V \to \Lambda V$, $\lambda \otimes v \mapsto lv$ (in M), is an isomorphism of Λ-modules.*

Proof. The polynomial algebra P_n is a commutative Noetherian domain, so any *maximal* (with respect to inclusion) element of the set of annihilators $\{\mathrm{ann}_{P_n}(v) \,|\, 0 \neq v \in M\}$ is a *prime* ideal. Fix such a prime ideal, say $\mathfrak{p} = \mathrm{ann}_{P_n}(v)$ for some $0 \neq v \in M$. Without loss of generality one can assume that $M = \mathcal{D}(P_n)v$. Then the $\mathcal{D}(P_n)$-module M is an epimorphic image of the $\mathcal{D}(P_n)$-module $\mathcal{D}(P_n)/\mathcal{D}(P_n)\mathfrak{p} \simeq \mathcal{D}(P_n) \otimes_{P_n} P_n/\mathfrak{p} = \cup_{i \geq 1} \mathrm{ann}(\mathfrak{p}^i)$. So, any element of M is annihilated by a power of the ideal of $\mathfrak{p}$. To prove the theorem we use induction on n.

The case $n = 1$. There are two cases: either $\mathfrak{p} = 0$ or otherwise $\mathfrak{p}$ is a maximal ideal of the polynomial algebra $P_1 := K[x]$. If $\mathfrak{p} = 0$ then $K[x]v \simeq K[x]$ is an isomorphism of $K[x]$-modules, and so it suffices to take $s = 1$ and $V = Kv$. If $\mathfrak{p} \neq 0$ then the ideal $\mathfrak{p}$ is generated by an irreducible polynomial of $K[x]$. Then the result follows from Lemma 6.3.

Suppose that $n > 1$ and the theorem is true for all $n' < n$. Now, we use a second downward induction on the Krull dimension $d = \mathrm{K.dim}\,(P_n/\mathfrak{p})$ of the algebra $P_n/\mathfrak{p}$ starting with $d = n$, i.e. $\mathfrak{p} = 0$. In this case, it suffices to take $\varepsilon = (1, \ldots, 1)$ and $V = Kv$, since $P_n v \simeq P_n$ is an isomorphism of P_n-modules.

Suppose now that $d < n$ and the result is true for all d' such that $d < d' \leq n$. The field of fractions $Q = \mathrm{Frac}(P_n/\mathfrak{p})$ of the domain $P_n/\mathfrak{p}$ has transcendence degree d over the field K, and it is generated by the elements $\overline{x}_i = x_i + \mathfrak{p}$, $i = 1, \ldots, n$. Up to order of the elements $\overline{x}_i$, we can assume that the elements $\overline{x}_1, \ldots, \overline{x}_d$ are algebraically independent over K, and so Q is the finite field extension of its subfield $Q_d := K(\overline{x}_1, \ldots, \overline{x}_d)$ of rational functions. Then

$P_n = P_d \otimes P_{n-d}$ where $P_d = K[x_1, \ldots, x_d]$ and $P_{n-d} = K[x_{d+1}, \ldots, x_n]$. Correspondingly, $\mathcal{D}(P_n) = \mathcal{D}(P_d) \otimes \mathcal{D}(P_{n-d})$ and the localization $Q_d \otimes_{P_d} \mathcal{D}(P_n) = Q_d \otimes_{P_d} \mathcal{D}(P_d) \otimes \mathcal{D}(P_{n-d}) \simeq \mathcal{D}(Q_d) \otimes \mathcal{D}(P_{n-d})$ of the algebra $\mathcal{D}(P_n)$ at $P_d \backslash \{0\}$ contains the subalgebra $Q_d \otimes \mathcal{D}(P_{n-d}) \simeq \mathcal{D}_{Q_d}(Q_d[x_{d+1}, \ldots, x_n])$ which is the ring of differential operators over the field Q_d of the polynomial algebra $Q_d[x_{d+1}, \ldots, x_n]$ in $n-d$ variables over the field Q_d. By the choice of the prime ideal $\mathfrak{p}$ and the elements $x_1, \ldots, x_d$, the $\mathcal{D}(P_n)$-module M is a *submodule* of its localization $Q_d \otimes_{P_d} M$ (use the fact that $\mathfrak{p} \cap P_d = 0$ and $M = \cup_{i \geq 1} \mathrm{ann}(\mathfrak{p}^i)$) which is a $Q_d \otimes_{P_d} \mathcal{D}(P_n)$-module, and, by restriction, it is a $\mathcal{D}_{Q_d}(Q_d[x_{d+1}, \ldots, x_n])$-module. Since $n - d < n$, by induction, one can find a subalgebra $\Lambda' = P_s(Q_d) \otimes_{Q_d} \bigotimes_{i=1}^{n-d-s} \Lambda_{-1}^{[p^{k_i}]}$ for some $k_i \geq 0$ and a finite dimensional Q_d-submodule of $\mathcal{D}_{Q_d}(Q_d[x_{d+1}, \ldots, x_n])$, say $\mathcal{V} = Q_d \otimes V$, of $Q_d \otimes_{P_d} M$ (where V is a finite dimensional K-submodule of M) such that $\dim_{Q_d}(\mathcal{V}) = \dim_K(V) \geq p^{k_1 + \cdots + k_{n-d-s}}$ and the natural map $\Lambda' \otimes_{Q_d} \mathcal{V} \to \Lambda' \mathcal{V}$ is an isomorphism of Λ'-modules. Let $\Lambda = P_d \otimes P_s \otimes \bigotimes_{i=1}^{n-d-s} \Lambda_{-1}^{[p^{k_i}]}$ (a subalgebra of $\mathcal{D}(P_n)$). Then the natural map $\Lambda \otimes V \to \Lambda V$ is an isomorphism. By induction, the proof now is complete. $\Box$

There is another proof of the inequality of Bernstein in prime characteristic.

Theorem 9.4 *Let K be an arbitrary field of characteristic $p > 0$. Then $\mathrm{Dim}(M) \geq n$ for each nonzero finitely generated $\mathcal{D}(P_n)$-module M.*

Proof. By Theorem 9.3, $\mathrm{Dim}(M) \geq \mathrm{Dim}(P_s \otimes \bigotimes_{i=1}^{n-s} \Lambda_{-1}^{[p^{k_i}]}) = n$ for some s and $k_i \geq 0$. $\Box$

The next theorem gives *explicitly* examples of sets of holonomic subalgebras with multiplicity 1 for the algebra $\mathcal{D}(P_n)$.

Theorem 9.5 *Let K be an arbitrary field of characteristic $p > 0$, $\mathcal{C} = \{$a subalgebra of $\mathcal{D}(P_n)$ of type $P_s \otimes \bigotimes_{i=1}^{n-s} \Lambda_{-1}^{[p^{k_i}]}$, $0 \leq s \leq n$, $k_i \geq 0\}$ and $\mathcal{C}' = \{\Lambda_\varepsilon \,|\, \varepsilon \in \{\pm 1\}^n\}$. Then the sets $\mathcal{C}$ and $\mathcal{C}'$ are holonomic sets of subalgebras with multiplicity 1 for the ring of differential operators $\mathcal{D}(P_n)$ (equipped with the canonical filtration).*

Proof. $\mathrm{Dim}(A) = n$ for all algebras A from $\mathcal{C} \cup \mathcal{C}'$. By Theorem 9.3, $\mathcal{C}$ is a holonomic set of subalgebras with multiplicity 1. Each algebra from the set $\mathcal{C}$ is a subalgebra of one of the algebras from the set $\mathcal{C}'$, and $\mathcal{C}' \subseteq \mathcal{C}$. Therefore, $\mathcal{C}'$ is a holonomic set of subalgebras with multiplicity 1 for the algebra $\mathcal{D}(P_n)$. $\Box$

Theorem 9.6 *Let K be an arbitrary field of characteristic $p > 0$. Then each holonomic $\mathcal{D}(P_n)$-module has finite length and its length does not exceed the multiplicity (i.e. the length of $M \leq \frac{l(M)}{n!}$, see Theorem 9.2).*

Proof. This follows from Theorems 9.2 and 9.5. $\Box$

Theorem 9.7 *Each holonomic $\mathcal{D}(P_n)$-module is cyclic.*

Proof. Repeat the characteristic zero proof which uses only that facts that each holonomic module has finite length and the ring of differential operators is simple and it is not an artinian module over itself. $\square$

An example of a cyclic non-holonomic $\mathcal{D}(P_n)$-module M with $\mathrm{Dim}(M) = n$. Consider the subalgebra $\Lambda = \Lambda_{-1}$ in $\mathcal{D}(P_1)$. Given an *infinite* sequence of natural numbers $\mathbf{k}$: $0 < k_1 < k_2 < \cdots$. Consider the cyclic Λ-module

$$M(\mathbf{k}) = \Lambda/\Lambda(\partial^{[j]} \,|\, j \in [1, p^{k_1}-1] \cup [p^{k_1}+1, p^{k_2}-1] \cup [p^{k_2}+1, p^{k_3}-1] \cup \ldots) \simeq K\overline{1} \oplus \oplus_{s\geq 1} K\partial^{[p^{k_s}]}\overline{1}$$

where $\overline{1}$ is the canonical generator for the Λ-module $M(\mathbf{k})$. Consider the filtration of standard type on $M(\mathbf{k})$ (with respect to the canonical filtration on $\mathcal{D}(P_1)$) $\{M(\mathbf{k})_i := \Lambda_i K\overline{1} = K\overline{1} \oplus K\partial^{[p^{k_1}]}\overline{1} \oplus \cdots \oplus K\partial^{[p^{k_s}]}\overline{1}\}$ where $s = s(i)$ satisfies $p^{k_s} \leq i < p^{k_{s+1}}$, and so $\dim_K(M(\mathbf{k})_i) = s(i) + 1$. For each $t \geq 1$, $\oplus_{t\geq s} K\partial^{[p^{k_s}]}\overline{1}$ is a submodule of $M(\mathbf{k})$, the corresponding factor module is denoted by $M(k_1, \ldots, k_{s-1}) = K\overline{1} \oplus K\partial^{[p^{k_1}]}\overline{1} \oplus \cdots \oplus K\partial^{[p^{k_{s-1}}]}\overline{1}$. In particular, $M(\emptyset) = K$.

The next lemma shows that the growth of the module $M(\mathbf{k})$ can be arbitrary slow.

Lemma 9.8 *For any non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ that takes infinitely many values and $f(0) = 1$, there exists a module $M(\mathbf{k})$ such that $\dim_K(M(\mathbf{k})_i) \leq f(i)$ for all $i \geq 0$ (for an arbitrary non-decreasing function f with $f(0) = 1$ there exists a Λ-module $M(k_1, \ldots, k_s)$ such that $\dim_K(M(k_1, \ldots, k_s)_i) \leq f(i)$ for all $i \geq 0$).*

Proof. One can easily find an infinite sequence of natural numbers $0 < k_1 < k_2 < \cdots$ satisfying the property that $\#\{j \,|\, p^{k_j} < f(i)\} \leq f(i)$ for all $i \geq 0$. $\square$

Proposition 9.9 *There exists a cyclic non-holonomic non-Noetherian $\mathcal{D}(P_n)$-module M such that* $\mathrm{Dim}(M) = n$.

Proof. Fix a Λ-module $M(\mathbf{k})$ from Lemma 9.8 which has zero growth, i.e. $\gamma(d_i) = 0$ where $d_i = \dim_K(M(\mathbf{k})_i)$. The Λ-module $M(\mathbf{k})$ is not a Noetherian module, hence the induced $\mathcal{D}(P_1)$-module $\mathcal{D}(P_1) \otimes_\Lambda M(\mathbf{k}) = P_1 \otimes M(\mathbf{k})$ is a cyclic non-Noetherian $\mathcal{D}(P_1)$-module. Since $\mathcal{D}(P_n) = \mathcal{D}(P_{n-1}) \otimes \mathcal{D}(P_1)$, the $\mathcal{D}(P_n)$-module

$$\mathcal{M}(\mathbf{k}) := P_{n-1} \otimes (\mathcal{D}(P_1) \otimes_\Lambda M(\mathbf{k})) \simeq P_n \otimes M(\mathbf{k})$$

is a cyclic non-Noetherian $\mathcal{D}(P_n)$-module. Let $\{\mathcal{M}_i\}$ be the filtration of standard type associated with the generating space $K\overline{1}$ for the $\mathcal{D}(P_n)$-module $\mathcal{M}(\mathbf{k})$ and the canonical filtration on $\mathcal{D}(P_n)$. Then

$$\dim_K(\mathcal{M}_i) = \binom{i+n}{n} + \binom{i+n-p^{k_1}}{n} + \cdots + \binom{i+n-p^{k_{d_i-1}}}{n} \leq d_i \binom{i+n}{n}.$$

It follows that $\mathrm{Dim}(\mathcal{M}(\mathbf{k})) = \gamma(\dim_K \mathcal{M}_i) \leq \gamma(d_i \binom{i+n}{n}) = \gamma(d_i) + n = n$.

Fix an arbitrary natural number l, then for all $i \gg 0$,

$$\begin{aligned}\dim_K \mathcal{M}_i &> \binom{i+n}{n} + \binom{i+n-p^{k_1}}{n} + \cdots + \binom{i+n-p^{k_l}}{n} \\ &\geq (l+1)\binom{i+n-p^{k_l}}{n} = \frac{(l+1)}{n!} i^n + \cdots .\end{aligned}$$

Therefore, $\mathrm{Dim}(\mathcal{M}(\mathbf{k})) = n$ and $\mathcal{M}(\mathbf{k})$ is not a holonomic $\mathcal{D}(P_n)$-module. $\square$

An example of a cyclic $\mathcal{D}(P_n)$-module M with $\mathrm{Dim}(M) = d$ for each $d \in [n, 2n]$. Given an ascending sequence b of positive real numbers $b_0 = 0 < b_1 < b_2 < \cdots$ with $\lim_{i\to\infty} b_i = \infty$ and a sequence s of positive real numbers $s_1, s_2, \ldots$. Consider a *continuous piecewise linear* function $f = f_{b,s} : \mathbb{R}_+ \to \mathbb{R}_+ := \{r \in \mathbb{R} \,|\, r \geq 0\}$ such that $f(0) = 1$ and on each interval $[b_{i-1}, b_i]$ it is a linear function with slope s_i. Then b and s are called the sequence of *breaking points* and *slopes* for $f_{b,s}$ respectively.

Let us explain an idea of the proof of Lemma 9.10 which is an essential step in proving Theorem 9.11. For any $r \in \mathbb{R}$ such that $0 < r < 1$, each linear function $ax + b$ with $a > 0$ grows faster then the function $y = x^r + 1$. The function $y = x^r + 1$ can be approximated by a function $f_{b,s}$ such that both functions have the *same* growth r and the graph of the function $f_{b,s}$ lies *below* the graph of the function $y = x^r + 1$. When the slopes tend to zero sufficiently fast then the restriction of the function $f_{b,s}$ to the set of natural numbers has the same growth. If we alter such a restriction at any subset of natural numbers such that the values at infinitely many breaking points remain unchanged, the new function from $\mathbb{N}$ to $\mathbb{R}_+$ is increasing, and its graph lies below the graph of $f_{b,s}$, then the altered function has growth r. Such an altered function will be the function that defines the growth of the Λ-module M_r from Lemma 9.10.

Lemma 9.10 *Let $\Lambda = K[\partial^{[1]}, \partial^{[2]}, \ldots,]$ and $r \in \mathbb{R}$, $0 < r < 1$.*

1. *There exists a cyclic Λ-module M_r such that $\mathrm{Dim}(M_r) = r$.*
2. *The $\mathcal{D}(P_1)$-module $\mathcal{M}_r := \mathcal{D}(P_1) \otimes_\Lambda M_r$ has dimension $\mathrm{Dim}(\mathcal{M}_r) = 1 + r$.*

Proof. 1. In this proof all functions are from $\mathbb{N}$ to $\mathbb{R}_+$. We are going to find an approximation of the function $y = x^r + 1$ by a function of the type $f = f_{b,s}$ where $s : s_1, p^{-k_1}, s_2, p^{-k_2}, s_3, p^{-k_3}, \ldots$ where $0 < k_1 < k_2 < \cdots$ and $b : b_0 = 0 < p^{k_1} < b_1 < p^{k_2} < b_2 < \cdots$. Fix a sufficiently big natural number, say k_1. Then s_1 is the slope of the linear function passing through the points $(0, 1)$ and $(p^{k_1}, 2)$, and so $f(p^{k_1}) = 2$. Let b_1 be the largest *natural* number of the form $i_1 p^{k_1}$ such that $i_1 \in \mathbb{N}$ and $f(b_1) < y(b_1)$. Then fix a sufficiently large natural number, say k_2, such that $b_1 < p^{k_2}$. Then s_2 is the slope of the linear function passing through the points $(b_1, f(b_1))$ and $(p^{k_2}, f(p^{k_2}) := f(b_1)+1)$. Let b_2 be the largest natural number of the type $i_2 p^{k_2}$ such that $i_2 \in \mathbb{N}$ and $p^{k_2} \leq b_2$ and $f(b_2) < y(b_2)$. We continue in a similar fashion. The graph of the function f lies below the graph of the function y. 'Sufficiently big' in the choices above means that $\lim_{i\to\infty} \frac{y(b_i) - f(b_i)}{f(b_i)} = 0$ (this can be easily achieved if the sequence $0 < k_1 < k_2 < \cdots$ grows sufficiently fast, this

condition guarantees that the values of the function f at the breaking points b_i are getting 'closer and closer' to the values of the function $y = x^r + 1$). Then $\gamma(f) = \gamma(y) = r$. For each $n \geq 1$, let $I_n = \{jp^{k_n}, 1 \leq j \leq i_n\}$, $I := \cup_{n\geq 1} I_n \cup \{0\}$, and $I' := \mathbb{N}\backslash I$. Consider the Λ-module $M_r := \Lambda/\Lambda(\partial^{[i]} \,|\, i \in I')$ and its filtration of standard type $\{M_{r,i}\}$ induced from the canonical filtration on the algebra Λ. Then $\dim_K(M_{r,j}) \leq f(j)$ for all $j \geq 0$, and $\dim_K(M_{r,b_\nu}) = f(b_\nu)$ for all $\nu \geq 1$. Therefore, $\mathrm{Dim}(M) := \gamma(\dim_K(M_{r,j})) = \gamma(f) = r$.

2. It follows from $\mathcal{M}_r = P_1 \otimes M_r$ that $\mathrm{Dim}(\mathcal{M}_r) = 1 + r$ since $\dim_K(P_{1,i}) = i + 1$ is a polynomial.

$\square$

Theorem 9.11 *Let K be a field of characteristic $p > 0$. Then $n \leq \mathrm{Dim}(M) \leq 2n$ for each nonzero finitely generated $\mathcal{D}(P_n)$-module M, and for each real number d such that $n \leq d \leq 2n$ there exists a cyclic $\mathcal{D}(P_n)$-module M such that $\mathrm{Dim}(M) = d$.*

Proof. Let M be a nonzero finitely generated $\mathcal{D}(P_n)$-module. Then $\mathrm{Dim}(M) \geq n$ by Theorem 9.4, and $\mathrm{Dim}(M) \leq \mathrm{Dim}(\mathcal{D}(P_n)) = 2n$. Therefore, $n \leq \mathrm{Dim}(M) \leq 2n$.

Given a real number d such that $n \leq d \leq 2n$. Then $d = n + s + r$ for some $s \in \mathbb{N}$ and $0 \leq r < 1$. If $r = 0$ then $\mathrm{Dim}(\mathcal{D}(P_s) \otimes P_{n-s}) = 2s + n - s = d$. If $r \neq 0$ then $\mathrm{Dim}(\mathcal{D}(P_s) \otimes P_{n-s-1} \otimes \mathcal{M}_r) = 2s + n - s - 1 + 1 + r = n + s + r = d$ where the $\mathcal{D}(P_1)$-module $\mathcal{M}_r$ is from Lemma 9.10. Obviously, the $\mathcal{D}(P_n)$-modules $\mathcal{D}(P_s) \otimes P_{n-s}$ and $\mathcal{D}(P_s) \otimes P_{n-s-1} \otimes \mathcal{M}_r$ are cyclic. $\square$

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